Electric Displacement ($\vec{D}$)

Dielectric Materials $\Rightarrow$ $\underbrace{\sigma_b = \vec{P}\cdot\hat{n} \ \ \& \ \ \rho_b = -\vec{\nabla}\cdot\vec{P}}_{\text{bound charges (immobile)}}$

Conductors $\Rightarrow$ $\underbrace{\sigma_f \ \& \ \rho_f}_{\text{free charges (mobile)}}$

$\therefore$ The total charge can be written

$$\rho = \rho_b + \rho_f$$

"Electric field produced by the total charge ρ"

the Gauss's Law : $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 = \frac{1}{\varepsilon_0}(\rho_b + \rho_f)$

$$= \frac{1}{\varepsilon_0}(-\vec{\nabla}\cdot\vec{P} + \rho_f).$$

$$\therefore \ \vec{\nabla}\cdot(\underbrace{\varepsilon_0\vec{E} + \vec{P}}) = \rho_f$$

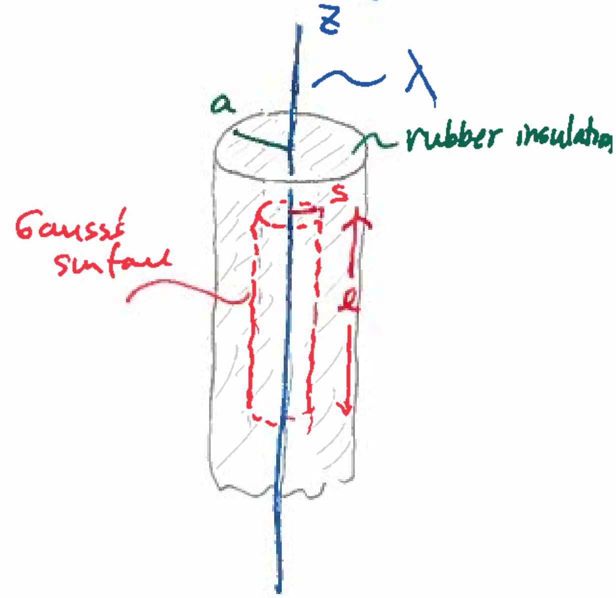
$\vec{D}$: electric displacement.

$\boxed{\vec{\nabla}\cdot\vec{D} = \rho_f}$ and $\boxed{\vec{D} = \varepsilon_0\vec{E} + \vec{P}}$.

" Gauss's law for free charge in a dielectric material "

$$\int_V \vec{\nabla}\cdot\vec{D}\, d\tau = \oint_S \vec{D}\cdot d\vec{a} = \int_V \rho_f\, d\tau = Q_{free}.$$

In high symmetry, we can calculate $\vec{D}$ from ρ_f using Gauss's Law



Ex Line charge of $\lambda \Rightarrow$ free charge.

$$\vec{\nabla}\cdot\vec{D} = \rho_{free}.$$

$$\therefore \oint_S \vec{D}\cdot d\vec{a} = \int_0^{\ell} \lambda\, dl$$

$$2\pi s \ell D = \lambda \ell$$

$$\therefore \vec{D} = \frac{\lambda}{2\pi s}\hat{s}$$

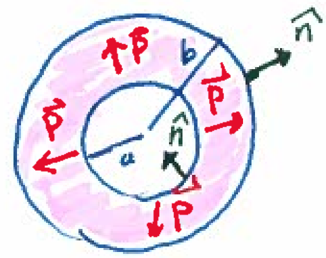
$\therefore$ For $s>a$ (no insulation, $\vec{P}=0$),

$$\vec{E} = (\vec{D}-\vec{P})/\varepsilon_0 = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s}$$

For $s<a$ (inside the insulation, $\vec{P}\neq 0$)

$$\vec{E} = (\vec{D}-\vec{P})/\varepsilon_0 .$$

HW 4.15. $\vec{P} = \frac{k}{r}\hat{r}$.



(a) Use $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 = \frac{1}{\varepsilon_0}(\rho_b+\rho_f)$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -k\vec{\nabla}\cdot\left(\frac{\hat{r}}{r}\right)$$

$$= -k\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r}\right)$$

$$= -\frac{k}{r^2}$$

$$\sigma_b = \vec{P}\cdot\hat{n} = \begin{cases} \vec{P}\cdot\hat{r} \quad (\text{at } r=b) = \frac{k}{b} \\ \vec{P}\cdot(-\hat{r}) \quad (\text{at } r=a) = -\frac{k}{a} \end{cases}$$

(i) $r<a$, $Q_{en}=0$ $\therefore$ $\vec{E}=0$

(ii) $a<r<b$, $Q_{en} = \sigma_b(4\pi a^2) + \frac{4}{3}\pi r^3\rho_b$

$$= \left(-\frac{k}{a}\right)4\pi a^2 + \int_a^r 4\pi r'^2\left(-\frac{k}{r'^2}\right)dr'$$

$$= -4\pi ka - 4\pi k(r-a) = -4\pi kr.$$

$\therefore$ $4\pi r^2 E = -4\pi kr/\varepsilon_0$. $\Rightarrow$ $\vec{E} = -\frac{k}{\varepsilon_0 r}\hat{r}$.

(iii) $r>b$. $Q_{en} = \sigma_b(r=a)(4\pi a^2) + \sigma_b(r=b)(4\pi b^2) + \int_a^b 4\pi(r')^2\cdot\left(-\frac{k}{r'^2}\right)dr.$

$$= -4\pi ka + 4\pi kb - 4\pi(b-a)k$$

$$= 0$$

$\therefore$ $\vec{E}=0$.

(b) Use $\vec{\nabla}\cdot\vec{D} = \rho_f$.

$\therefore \quad Q_f = 0$ (no free charge).

$\therefore \quad \vec{D} = \varepsilon_0\vec{E} + \vec{P} = 0 \qquad \therefore \quad \vec{E} = -\vec{P}/\varepsilon_0$.

(i) For $r<a$ and $r>b$, $\vec{P}=0 \Rightarrow \vec{E}=0$

For $a<r<b$, $\quad \vec{E} = -\frac{k}{\varepsilon_0}\frac{\hat{r}}{r}$

$\boxed{HW}$ 4.16.

- $\vec{E}$, $\vec{D}$, and Gauss's Law.

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \quad \rightarrow \quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{r}')}{\eta^2}\hat{\eta}\, d\tau' \quad \text{(Coulomb's Law)}.$$

$$\vec{\nabla}\cdot\vec{D} = \rho_f \quad \overset{?}{\Rightarrow} \quad \vec{D} \neq \frac{1}{4\pi}\int_V \frac{\rho_f(\vec{r}')}{\eta^2}\hat{\eta}\, d\tau'$$

In $\vec{E}$, there is another condition required to satisfy the Coulomb's law: $\vec{\nabla}\times\vec{E} = 0$

With $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ & $\vec{\nabla}\times\vec{E} = 0$, $\vec{E}$ is uniquely determined!

However, $\vec{\nabla}\times\vec{D} = \vec{\nabla}\times(\varepsilon_0\vec{E} + \vec{P}) = \varepsilon_0\cancelto{0}{\vec{\nabla}\times\vec{E}} + \vec{\nabla}\times\vec{P} = \vec{\nabla}\times\vec{P} \neq 0$.
in general!

Therefore, $\vec{\nabla}\times\vec{D} \neq 0$ and there is no corresponding potential for $\vec{D}$.

- Boundary Conditions (review how we extracted B.C. for $\vec{E}$).

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}; \qquad E^{\parallel}_{above} - E^{\parallel}_{below} = 0.$$

But

$$D^{\perp}_{above} - D^{\perp}_{above} = \frac{\sigma_f}{\varepsilon_0}; \qquad D^{\parallel}_{above} - D^{\parallel}_{below} = P^{\parallel}_{above} - P^{\parallel}_{below}$$

▶ Susceptibility (χ_e), Permittivity (ε), Dielectric Const. (ε_r).

$$\vec{P} = \alpha \vec{E} = \varepsilon_0 \boxed{\chi_e} \vec{E} \qquad (\vec{P} \parallel \vec{E}) \Rightarrow \text{Linear Dielectrics.}$$

polarizability

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}(\rho_f + \rho_b).$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (1+\chi_e)\vec{E} = \boxed{\varepsilon}\vec{E} \qquad \vec{D} = \varepsilon \vec{E}$$

Vacuum permittivity. Permittivity.

where $\varepsilon = \varepsilon_0 (1+\chi_e)$,

$\varepsilon_r \equiv \varepsilon / \varepsilon_0 = 1 + \chi_e$: Relative permittivity / Dielectric Const.

	Vacuum	He	Ne	H_2O (v)	Diamond	Ice (H_2O)
ε_r	1	1.000065	1.00013	1.00589	5.7–5.9	104.

EX $V(r=0) = ?$ need to know $\vec{E}$

$\vec{D} = \varepsilon \vec{E}$ (linear dielectrics).

(i) We know $\vec{E} = 0$ for $r < a$.

(ii) $\vec{\nabla}\cdot\vec{D} = \rho_f$

$\therefore 4\pi r^2 D = Q$

$$\therefore \vec{D} = \frac{Q}{4\pi r^2}\hat{r} \Rightarrow \vec{E} = \begin{cases} \dfrac{Q}{4\pi\varepsilon r^2}\hat{r} & (a<r<b) \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r} & (r>b) \end{cases}$$

Smaller than this ($\varepsilon > \varepsilon_0$)

$$\therefore V(0) = -\int_{\infty}^{0} \vec{E}\cdot d\vec{r} = -\int_{\infty}^{b} \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}dr - \int_{b}^{a} \frac{1}{4\pi\varepsilon}\frac{Q}{r^2}dr - \int_{a}^{0} 0\, dr$$

$$= \frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b}\right).$$

And. $\vec{P} = \varepsilon_0 \chi_e \vec{E} = \dfrac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon_0 r^2}\hat{r}$

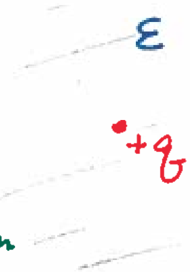
$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$

$$\sigma_b(r=a) = \vec{P}\cdot\hat{n} = \vec{P}\cdot(-\hat{r}) = -\frac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon_0 a^2}$$

$$\sigma_b(r=b) = \vec{P}\cdot\hat{n} = \vec{P}\cdot(+\hat{r}) = +\frac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon_0 b^2}.$$

(*) When the whole space where $\vec{E}$ is present is filled with a linear dielectric medium with ε, the resulting electric field can be obtained by replacing ε_0 with ε in $\vec{E}$ evaluated in vacuum (free space).

$\overline{\text{EX}}$ Electric field from a point charge $+q$ placed at the origin inside a $\underset{\text{linear}}{\smile}$ dielectric medium w/ ε.

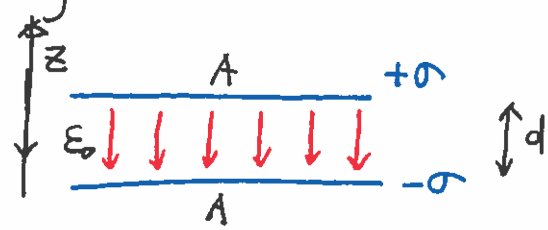


$$\underbrace{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}}_{\text{free space}} \Rightarrow \underbrace{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon}\frac{q}{r^2}\hat{r}}_{\text{dielectric medium}}$$

$\overline{\text{EX}}$ $\vec{E}$ between two paralled conducting plates

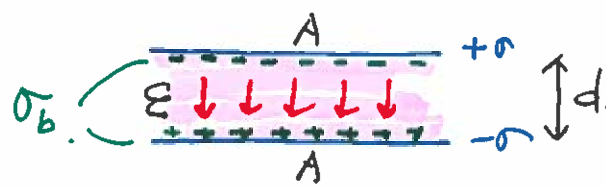
$\vec{E}_v = \frac{\sigma}{\varepsilon_0}(-\hat{z})$ in vacuum

$\vec{E}_d = \frac{\sigma}{\varepsilon}(-\hat{z})$ in dielectric M.



$|\vec{E}_v| > |\vec{E}_d|$.

reduction due to σ_b



$\therefore\ C_v = \frac{\varepsilon_0 A}{d}$ but $C_d = \frac{\varepsilon A}{d}$

$\therefore\ C_d = \frac{\varepsilon}{\varepsilon_0}\frac{\varepsilon_0 A}{d} = \varepsilon_r C_v$

$\boxed{\text{HW}}$ 4.18, 4.21.