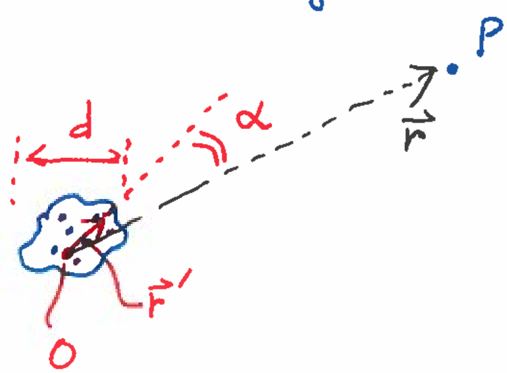
Multipole Expansion.

$V(\vec{r})$ or $\vec{E}(\vec{r})$ from a localized charge distribution.
for $r \gg d$ where d is typical size of the localized charge.

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{r}') \cdot \frac{1}{\mathscr{r}}\, d\tau'$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}'.$$

$$Q = \int_V \rho(\vec{r}')\, d\tau'$$

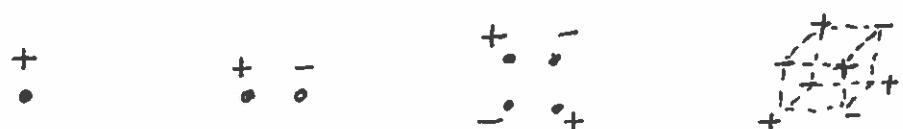


For $r \gg d$,

$$V(\vec{r}) \approx \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r}$$ ~ behaves like a pt. charge for $r \gg d$.

However, if $Q = 0$. but $\rho(\vec{r}) \neq 0$, then.

$$V(\vec{r}) \sim \frac{1}{r^n} \quad (n \geq 2.) \quad \text{or } E \sim \frac{1}{r^{n+1}} \quad (n \geq 2).$$



	$Q = +$	$Q = 0$	$Q = 0$	$Q = 0$	
$V \sim$	$\frac{1}{r}$	$\frac{1}{r^2}$	$\frac{1}{r^3}$	$\frac{1}{r^4}$	leading term
$E \sim$	$\frac{1}{r^2}$	$\frac{1}{r^3}$	$\frac{1}{r^4}$	$\frac{1}{r^5}$	
	Monopole	Dipole	Quadrupole	Octopole.	

Taylor Expansion! $r \gg d$.

$$\mathscr{r}^2 = r^2 + (r')^2 - 2rr'\cos\alpha \qquad \cos\alpha = \angle(\vec{r}', \vec{r}).$$

Pay attention to the location of the origin!

$r' \lesssim d \quad \therefore \; r \gg r'$

$$\mathscr{r}^2 = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right] = r^2(1+\epsilon)$$

where $\epsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$.

$\therefore\ \mathscr{r} = r\sqrt{1+\epsilon}$. and

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\cdot\frac{1}{\sqrt{1+\epsilon}} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right)$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \cdots\right]$$

$$= \frac{1}{r}\Big[1 + \left(\frac{r'}{r}\right)\cos\alpha - \frac{1}{2}\left(\frac{r'}{r}\right)^2 + \frac{3}{\cancel{8}_2}\cdot\cancel{4}^{1}\cos^2\alpha\left(\frac{r'}{r}\right)^2 + \frac{3}{\cancel{8}_2}(\cancel{-4})\left(\frac{r'}{r}\right)^3\cos\alpha + \frac{3}{8}\left(\frac{r'}{r}\right)^4$$

$$- \frac{5}{\cancel{16}_2}\left(\frac{r'}{r}\right)^3(\cancel{-8})\cos^3\alpha - \frac{5}{\cancel{16}_4}(3\cdot\cancel{4})\left(\frac{r'}{r}\right)^4\cos^2\alpha - \frac{5}{\cancel{16}_8}(\cancel{-2})\cdot 3\left(\frac{r'}{r}\right)^5\cos\alpha - \frac{5}{16}\left(\frac{r'}{r}\right)^6 + \cdots\Big]$$

$$= \frac{1}{r}\left[\underbrace{1}_{P_0(\cos\alpha)} + \underbrace{(\cos\alpha)}_{P_1(\cos\alpha)}\left(\frac{r'}{r}\right) + \underbrace{\left(\frac{3\cos^2\alpha - 1}{2}\right)}_{P_2(\cos\alpha)}\left(\frac{r'}{r}\right)^2 + \underbrace{\left(\frac{5\cos^3\alpha - 3\cos\alpha}{2}\right)}_{P_3(\cos\alpha)}\left(\frac{r'}{r}\right)^3 + \cdots\right]$$

$$= \frac{1}{r}\left[P_0(\cos\alpha) + \left(\frac{r'}{r}\right)^1 P_1(\cos\alpha) + \left(\frac{r'}{r}\right)^2 P_2(\cos\alpha) + \left(\frac{r'}{r}\right)^3 P_3(\cos\alpha) + \cdots\right]$$

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha).$$

Multipole Expansion of V — "Legendre Polyn.'s"

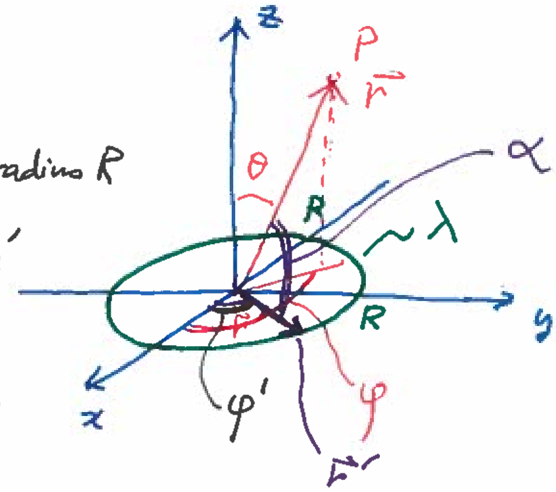
$$\therefore\ \boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int_V (r')^n P_n(\cos\alpha)\,\rho(\vec{r}\,')\,d\tau'}$$

$$= \frac{1}{4\pi\epsilon_0}\Big[\underbrace{\frac{1}{r}\int_V \rho(\vec{r}\,')\,d\tau'}_{\text{monopole}} + \underbrace{\frac{1}{r^2}\int_V r'\rho(\vec{r}\,')P_1(\cos\alpha)\,d\tau'}_{\text{dipole}}$$

$$+ \underbrace{\frac{1}{r^3}\int (r')^2\rho(\vec{r}\,')P_2(\cos\alpha)\,d\tau'}_{\text{quadrupole}} + \cdots\Big]$$

<u>EX</u> Prob 3.28

Uniform ring charge with λ of radius R



$$V(\vec r) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_V (r')^n P_n(\cos\alpha)\rho(\vec r')d\tau'$$

$$\vec r = (r,\theta,\varphi) = r\sin\theta\cos\varphi\,\hat x + r\sin\theta\sin\varphi\,\hat y + r\cos\theta\,\hat z$$

$$\vec r' = (R,\tfrac{\pi}{2},\varphi') = R\cos\varphi'\,\hat x + R\sin\varphi'\,\hat y$$

$$\vec r\cdot\vec r' = r r'\cos\alpha = rR\,(\underbrace{\sin\theta\cos\varphi\cos\varphi' + \sin\theta\sin\varphi\sin\varphi'}_{\cos\alpha}).$$

$$\int_V \rho(\vec r')d\tau' \longrightarrow \int_C \lambda(\vec r')dl' = \int_0^{2\pi}\lambda R\,d\varphi'$$

For $n=0$ (monopole)

$$V_0 = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int_0^{2\pi}(1)\lambda R\,d\varphi' = \frac{\lambda R}{2\varepsilon_0}\left(\frac{1}{r}\right).$$

For $n=1$ (dipole)

$$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int_0^{2\pi} \underset{=R}{r'}\,P_1(\cos\alpha)\lambda R\,d\varphi'$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r^2}\right)\lambda R^2\int_0^{2\pi}(\sin\theta\cos\varphi\underline{\underline{\cos\varphi'}} + \sin\theta\sin\varphi\underline{\underline{\sin\varphi'}})d\varphi'$$

$= 0$ (no dipole)

For $n=2$

$$V_2 = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\int_0^{2\pi}(r')^2 P_2(\cos\alpha)\lambda R\,d\varphi'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda R^3}{r^3}\int_0^{2\pi}\frac{1}{2}(3\cos^2\alpha - 1)\,d\varphi' = \frac{1}{4\pi\varepsilon_0}\left(\frac{\lambda R^3}{2r^3}\right)\int_0^{2\pi}\{3\sin^2\theta(\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi')^2 - 1\}\,d\varphi$$

$$* \quad \int_0^{2\pi}\sin^2\theta\,d\theta = \int_0^{2\pi}\cos^2\theta\,d\theta = \pi \quad\text{and}\quad \int_0^{2\pi}\sin\theta\cos\theta\,d\theta = 0.$$

$$\therefore\ V_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{\lambda R^3}{2r^3}\right)\left[3\sin^2\theta\,(\underbrace{\pi\cos^2\varphi+\pi\sin^2\varphi}_{\pi})-2\pi\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{\lambda R^3}{2r^3}\right)\pi\left(3\sin^2\theta-2\right)$$

$$= \frac{1}{8\varepsilon_0}\left(\frac{\lambda R^3}{r^3}\right)\left(3\sin^2\theta-2\right)$$

$$= \frac{-\lambda}{8\varepsilon_0}\left(\frac{R}{r}\right)^3\left(3\cos^2\theta-1\right) = -\frac{\lambda}{4\varepsilon_0}\left(\frac{R}{r}\right)^3 P_2(\cos\theta).$$

$\boxed{\text{HW}}$ 3.27

- Dipole moment.

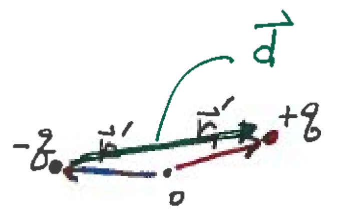
$\vec{p} \equiv \int \vec{r}\,'\rho(\vec{r}\,')\,d\tau'$ for continuous charge distribution

Then $V_{dipole} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int r'\rho(\vec{r}\,')P_1(\cos\alpha)\,d\tau'$

$= \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$

For a collection of pt charges q_i's at $\vec{r}_i\,'$

$\vec{p} = \sum_{i=1}^{n}\vec{r}_i\,' q_i$



$|\vec{p}| \approx qd$

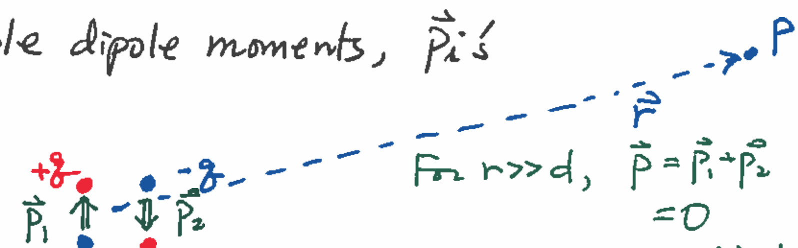
Perfect dipole is a mathematical object $d\to 0$ while $qd = p \neq 0$.

In reality, if $|\vec{d}| \ll \vec{r}$, $\vec{p} = q\vec{d}$ can work as a perfect dipole!

$\vec{p} = \vec{r}_+'q + \vec{r}_-'(-q)$
$= (\vec{r}_+' - \vec{r}_-')q$
$= q\vec{d}$

One can have multiple dipole moments, $\vec{p}_i$'s

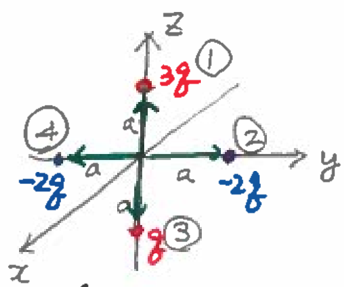
$\therefore\ \vec{p} = \sum_{i=1}^{n}\vec{p}_i$



For $r \gg d$, $\vec{p} = \vec{p}_1 + \vec{p}_2 = 0$

* But we have a quadrupole moment!

$\overline{\text{EX}}$ What is $V(\vec{r})$ for $r \gg a$



$Q = \sum_i q_i = 0$ (no monopole).

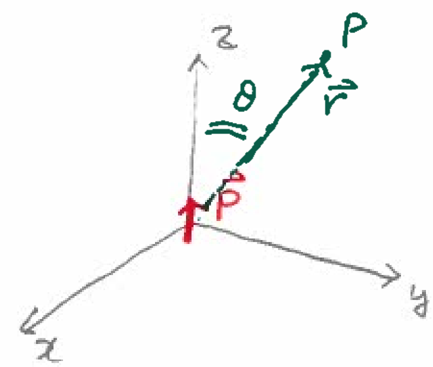
$$\vec{p} = \sum_{i=1}^{4} \vec{r}_i' q_i$$

$$= \{3q(+\hat{z}) + (-2q)(+\hat{y}) + q(-\hat{z}) - 2q(-\hat{y})\} a$$

$$= 2qa\hat{z}$$

$\boxed{\text{HW}}$ 3.30

◎ Dipole field $\vec{E}_{dip}$



$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2} \quad \text{(see the geometry)}$$

$$(\vec{p} = p\hat{z}) \quad = \frac{p}{4\pi\varepsilon_0} \frac{\cos\theta}{r^2}$$

$$\therefore \vec{E}_{dip} = -\vec{\nabla} V_{dip}$$

$$= -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\hat{\varphi}$$

$$= \frac{2p\cos\theta}{4\pi\varepsilon_0}\left(\frac{1}{r^3}\right)\hat{r} + \frac{p\sin\theta}{4\pi\varepsilon_0}\left(\frac{1}{r^3}\right)\hat{\theta} + 0 .$$

$$\vec{E}_{dip}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right).$$

For $\theta = 0$ $\quad \vec{E}_{dip} = \frac{p}{2\pi\varepsilon_0}\left(\frac{1}{r^3}\right)\hat{r}$

For $\theta = \frac{\pi}{2}$ $\quad \vec{E}_{dip} = \frac{p}{4\pi\varepsilon_0}\left(\frac{1}{r^3}\right)\hat{\theta}$

} azimuthal sym. no φ-dep.

$\boxed{\text{HW}}$ 3.35, 3.36, 3.52* if you want to (a) (b) (c) only