$\frac{1}{h_1 h_2 h_3}$ $\frac{h_2 h_3}{h_1}$

$\boxed{III\,2}$ 1

◎ $\nabla^2 V = 0$ in (r, θ, φ) with azimuthal symm. $V = V(r, \theta)$ no φ.

$$\nabla^2 V(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \cancel{\frac{1}{r^2\sin\theta}\frac{\partial^2 V}{\partial\varphi^2}}^{\,0} = 0.$$

$$\therefore \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0.$$

$V(r,\theta) = R(r)\Theta(\theta)$ ⇒ separation of variables

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)}_{\ell(\ell+1)} + \underbrace{\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)}_{= -\ell(\ell+1)} = 0. \quad (\text{const}).$$

(i) $\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \ell(\ell+1) \Rightarrow \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \ell(\ell+1)R$.

2nd order ordinary diff. eq.

Expanding the derivative

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \ell(\ell+1)R = 0. \quad \text{———} \quad (1)$$

* The order of derivatives ~~are~~ is compensated by multiplying the same order of r^n! ⇒ R should be $\sim r^n$ ~~[illegible]~~

Try $R \sim r^n$, for $n \geq 1$

$$r^2 n(n-1) r^{n-2} + 2r n r^{n-1} - \ell(\ell+1) r^n = 0.$$

$$\{n(n-1) + 2n - \ell(\ell+1)\} r^n = 0 \quad \Rightarrow \quad n = \ell.$$

$$\therefore \quad R \sim r^\ell$$

Try $R \sim r^{-n}$ for $n \geq 1$

$$r^2 (-n)(-n-1) r^{-n-2} + 2r(-n) r^{-n-1} - \ell(\ell+1) r^{-n} = 0$$

$$\{n(n+1) - 2n - \ell(\ell+1)\} r^{-n} = 0 \quad \Rightarrow \quad n = \ell+1$$

$$\therefore \quad R \sim r^{-(\ell+1)}.$$

The general solution should be

$$R_{\underline{\underline{\ell}}}(r) = A r^{\ell} + \frac{B}{r^{\ell+1}}.$$

index.

(ii) $$\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -\ell(\ell+1)\sin\theta\, \Theta. \qquad \text{———} (2)$$

$\Theta(\theta) = P_\ell(\cos\theta)$: Legendre Polynomials.

Let's $x \equiv \cos\theta \quad (-1 \le x \le 1)$

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell.$$

for $\ell = 0, 1, 2, \cdots$

$$\boxed{\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= (3x^2-1)/2 \\ P_3(x) &= (5x^3-3x)/2 \\ P_4(x) &= (35x^4-30x^2+3)/8 \\ P_5(x) &= (63x^5-70x^3+15x)/8 \end{aligned}}$$

* $P_\ell(-x) = (-1)^\ell P_\ell(x)$.
* $P_\ell(x)$ has x^ℓ the highest order term.
* $P_\ell(x) = 0$ has ℓ solutions.
* $P_\ell(x)$ is one sol. for Eq (2)

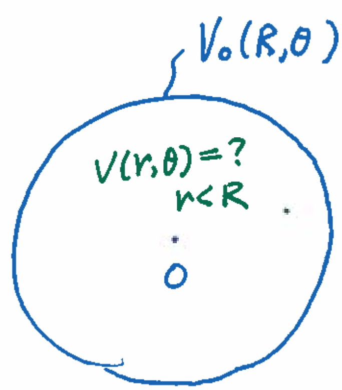
2^{nd} order D.E.. There should be other solutions. But they diverges at $\theta = 0$, or π.

**

$$\therefore \boxed{V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)} \text{ with azimuthal sym.}$$

P_ℓ's are orthogonal functions.

$$\int_{-1}^{+1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell,\ell'}$$



Ex

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta).$$

(i) $V(r\to 0)$ should be ~~finite~~ (no singular).

$\therefore B_\ell = 0$

(ii) $$V(R,\theta) = V_0(\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta).$$

One can determine A_ℓ using the orthogonality of $P_\ell(x)$.

$$\int_{-1}^{+1} P_{\ell'}(x)\, V_0 \, dx = \int_{-1}^{+1} P_{\ell'}(x) \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(x)\, dx$$

$$= \sum_{\ell=0}^{\infty} R^\ell A_\ell \int_{-1}^{+1} P_{\ell'}(x) P_\ell(x)\, dx$$

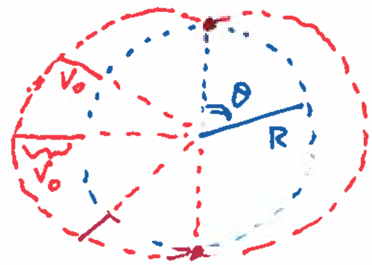
$$= \sum_{\ell=0}^{\infty} R^\ell A_\ell \cdot \frac{2}{2\ell+1} \delta_{\ell,\ell'}$$

$$= \frac{2}{2\ell'+1} A_{\ell'} R^{\ell'}$$

$$\therefore \quad A_{\ell'} = \frac{2\ell'+1}{2R^{\ell'}} \int_{-1}^{+1} P_{\ell'}(x)\, V_0 \, dx = \frac{2\ell'+1}{2R^{\ell'}} \int_0^\pi P_{\ell'}(\cos\theta)\, V_0(\theta) \cancel{d\cos} \sin\theta\, d\theta .$$

For $V_0(\theta) = k \sin^2\left(\frac{\theta}{2}\right)$.

$$= \frac{k}{2}\Big(\underbrace{1}_{P_0} - \underbrace{\cos\theta}_{P_1}\Big)$$



$$A_\ell = \frac{2\ell+1}{2R^\ell} \int_{-1}^{+1} \frac{k}{2}(1-x) P_\ell(x)\, dx .$$

$$= \frac{2\ell+1}{2R^\ell}\left(\frac{k}{2}\right) \int_{-1}^{+1} (P_0 P_\ell - P_1 P_\ell)\, dx .$$

$$= \frac{2\ell+1}{2R^\ell}\left(\frac{k}{2}\right) \cdot \left[2\delta_{\ell,0} - \frac{2}{3}\delta_{\ell,1} \right]$$

$$\therefore \quad A_0 = \frac{k}{2R^0} = \frac{k}{2}, \qquad A_1 = \frac{k}{2R}$$

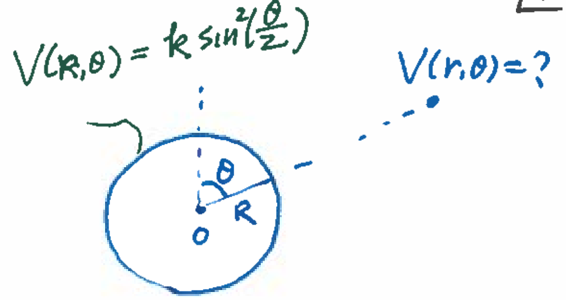
all other A_ℓ's $= 0$

$$\therefore \quad V(r,\theta) = \frac{k}{2}\left\{ P_0(\cos\theta) - \frac{r}{R} P_1(\cos\theta) \right\}$$

$$= \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta \right).$$

EX

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l \frac{1}{r^{l+1}} \right) P_l(\cos\theta).$$



(i) $V(r \to \infty, \theta) = 0.$

(ii) $V(r=R, \theta) = k \sin^2 \frac{\theta}{2}.$

(i) $A_l = 0 \quad (\because \lim_{r\to\infty} r^l = \infty).$

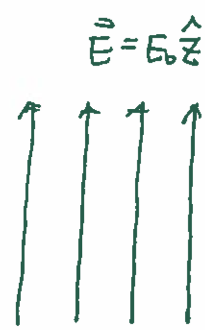
(ii) $V(R,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = k \sin^2 \frac{\theta}{2} = \frac{k}{2}\left(P_0(\cos\theta) - P_1(\cos\theta)\right).$

$$\therefore \int_0^{\pi} V(R,\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} \underbrace{\int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta}_{\frac{2}{2l'+1}\delta_{l,l'}}.$$

$$\therefore B_l = \frac{2l+1}{2} R^{l+1} \int_0^{\pi} \underbrace{V(R,\theta)}_{\frac{1}{2}k\{P_0 - P_1\}} P_l(\cos\theta) \sin\theta \, d\theta.$$

* Finish the calculation !

EX $\quad V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta).$



(i) $V(r=R, \theta) = 0$

(ii) For $r \gg R$, $\quad \vec{E} \to E_0 \hat{z}.$

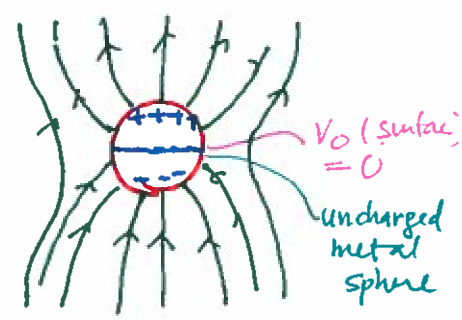
$\therefore V \to -E_0 z + \text{const}.$

$\quad\;\; \to -E_0 r \cos\theta$

this should be 0 for all
$\because V=0$
$r=R$ $(\theta = \frac{\pi}{2})$

(i). $V(R,\theta) = \sum_{l=0}^{\infty} \underbrace{\left(A_l R^l + \frac{B_l}{R^{l+1}} \right)} P_l(\cos\theta) = 0.$

should be zero for each l.



$$A_l R^l + \frac{B_l}{R^{l+1}} = 0 \Rightarrow B_l = -A_l R^{2l+1}.$$

$$\therefore V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta)$$

(ii) $r \gg R \Rightarrow \frac{R^{2l+1}}{r^{l+1}} \to 0.$

$\therefore$

$$V(r \gg R, \theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) = -E_0 r \underbrace{\cos\theta}_{P_1(\cos\theta)}.$$

$\therefore$ $A_1 = -E_0$ and $A_\ell = 0$ for $\ell \geq 2$

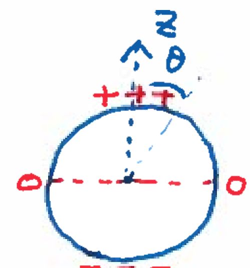
Finally,

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta.$$

$$= \underbrace{-E_0 r\cos\theta}_{\text{from } \vec{E} = E_0 \hat{z}} + \underbrace{E_0 \frac{R^3}{r^2}\cos\theta}_{\text{from } \vec{E} \text{ induced charge}}.$$

$$\sigma(\theta) = -\varepsilon_0 \frac{\partial V}{\partial n} = -\varepsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R}$$

$$= \varepsilon_0\left(E_0\cos\theta + 2E_0\frac{R^3}{R^3}\cos\theta\right) = 3\varepsilon_0 E_0 \cos\theta$$



EX

* For uniform σ, it is a Gauss's law problem and simple!

(i) $V \to 0$ for $r \gg R$

V should be regular for $r \to 0$ $(< R)$.

(ii) $V(R,\theta)$: prober B.C.

(i) For $r > R$, $\quad V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta).$

$A_\ell = 0. \Rightarrow V_>(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$

But for $r < R$, ~~$\forall \ell$~~ $B_\ell = 0 \Rightarrow V_<(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta).$

" $\vec{E}$ is discontinuous at $r = R$ but V should be continuous."

$$\therefore \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta).$$

$$\therefore \boxed{B_\ell = A_\ell R^{2\ell+1}} \quad \text{———} \; (*)$$

And also

$$\left.\frac{\partial V_>}{\partial r} - \frac{\partial V_<}{\partial r}\right|_{r=R} = -\frac{1}{\varepsilon_0}\sigma(\theta) .$$

$V_> = V(r>R,\theta)$; $V_< = V(r<R,\theta)$.

$$-\sum_{\ell=0}^{\infty}(\ell+1)\frac{B_\ell}{R^{\ell+2}}P_\ell(\cos\theta) - \sum_{\ell=0}^{\infty}\ell A_\ell R^{\ell-1}P_\ell(\cos\theta) = -\frac{1}{\varepsilon_0}\sigma(\theta)$$

Using ⊛,

$$-\sum_{\ell=0}^{\infty}(\ell+1)A_\ell R^{\ell-1}P_\ell(\cos\theta) - \sum_{\ell=0}^{\infty}\ell A_\ell R^{\ell-1}P_\ell(\cos\theta) = -\frac{1}{\varepsilon_0}\sigma(\theta) .$$

$$\therefore \sum_{\ell=0}^{\infty}(2\ell+1)A_\ell R^{\ell-1}P_\ell(\cos\theta) = +\frac{1}{\varepsilon_0}\sigma(\theta) .$$

Again using the orthogonality condition, we can get A_ℓ's !

$$\sum_{\ell'=0}^{\infty}\int_0^\pi(2\ell'+1)A_{\ell'}R^{\ell'-1}P_{\ell'}(\cos\theta)P_\ell(\cos\theta)\sin\theta\, d\theta = +\frac{1}{\varepsilon_0}\int_0^\pi\sigma(\theta)P_\ell(\cos\theta)\sin\theta\, d\theta .$$

$$\sum_{\ell'=0}^{\infty}(2\ell'+1)A_{\ell'}R^{\ell'-1}\underbrace{\int_0^\pi P_{\ell'}(\cos\theta)P_\ell(\cos\theta)\sin\theta\, d\theta}_{\frac{2}{2\ell+1}\delta_{\ell,\ell'}} = +\frac{1}{\varepsilon_0}\int_0^\pi\sigma(\theta)P_\ell(\cos\theta)\sin\theta\, d\theta$$

$$\cancel{(2\ell+1)}A_\ell R^{\ell-1}\cdot\frac{2}{\cancel{2\ell+1}} = +\frac{1}{\varepsilon_0}\int_0^\pi\sigma(\theta)P_\ell(\cos\theta)\sin\theta\, d\theta .$$

$$\therefore A_\ell = \frac{1}{2\varepsilon_0 R^{\ell-1}}\cdot\int_0^\pi\sigma(\theta)P_\ell(\cos\theta)\sin\theta\, d\theta .$$

For example, if $\sigma(\theta) = k\cos\theta = kP_1(\cos\theta)$.

$$\therefore A_\ell = \frac{1}{2\varepsilon_0 R^{\ell-1}}k\underbrace{\int_0^\pi P_1(\cos\theta)P_\ell\sin\theta\, d\theta}_{\frac{2}{3}\delta_{\ell,1}} = \begin{cases}\frac{k}{3\varepsilon_0} & \text{for } \ell=1\\ 0 & \text{for all other } \ell\text{'s}\end{cases}$$

$\therefore$ For $r<R$, $V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) = \frac{k}{2\varepsilon_0} r P_1(\cos\theta)$

$= \frac{k}{3\varepsilon_0} r \cos\theta$.

For $r>R$, $V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell \frac{R^{2\ell+1}}{r^{\ell+1}} P_\ell(\cos\theta) = \frac{k}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta$.

$\boxed{\text{HW}}$ 3.18, 3.19, 3.23, & 3.25.

(s,φ,z) with cylindrical symm. $\Rightarrow$ no z-dependence!

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial\varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

$$V(s,\varphi,z) = S(s)\,\Phi(\varphi)$$

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial S\Phi}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 S\Phi}{\partial\varphi^2} = \Phi\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) + \frac{S}{s^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0.$$

$\therefore$ Multiply $\frac{s^2}{S\Phi}$ on both sides.

$$\underbrace{\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right)}_{k^2} + \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}}_{-k^2} = 0.$$

$$s\frac{d}{ds}\left(s\frac{dS}{ds}\right) = k^2 S \quad (1) \quad \Rightarrow S \sim s^n \quad (n\neq 0).$$

$\therefore$ $s\frac{d}{ds}\left(s\cdot n s^{n-1}\right) = n^2 s^n = k^2 s^n$ $\quad\therefore\quad n = \pm k$.

$\therefore$ $S = A s^n + B s^{-n}$ $\cdot$ $(n\geq 1)$

for $n=0$, $\frac{d}{ds}\left(s\frac{dS}{ds}\right) = 0 \Rightarrow s\frac{dS}{ds} = C$ $\quad\therefore S = C\ln s + D$.

then $\frac{d^2\Phi}{d\varphi^2} = 0 \Rightarrow \Phi(\varphi) = E\varphi + F$

* $E=0$ because $\varphi = \varphi + m2\pi$, $E\varphi$ solution is not allowed!

Therefore,

$$V(s,\varphi) = C \ln s + D + \sum_{n=1}^{\infty} \Big[s^n (a_n \cos n\varphi + b_n \sin n\varphi) + s^{-n} (C_n \cos n\varphi + D_n \sin n\varphi) \Big].$$