II 11

II

Solving Laplace's Eq: Technique II (Separation of Variables)

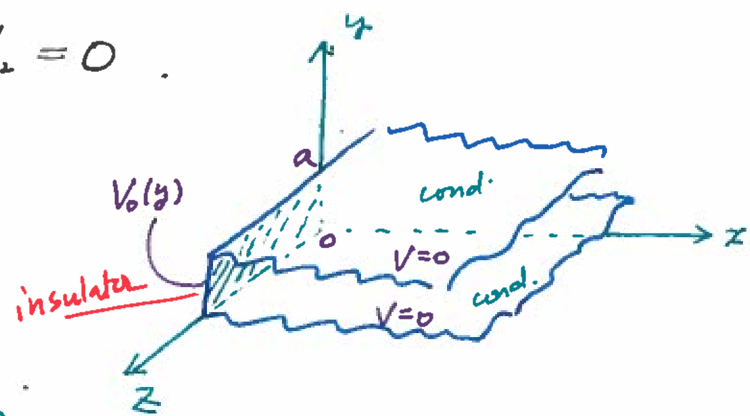
$$V(x,y,z) = X(x)Y(y)Z(z)$$
$$V(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi)$$
$$V(s,\varphi,z) = S(s)\Phi(\varphi)Z(z).$$

① (x,y,z).

$$\nabla^2 V(x,y,z) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

B.C.'s

(i) $V=0$ for $y=0$
(ii) $V=0$ for $y=a$
(iii) $V=V_0(y)$ for $x=0$
(iv) $V\to 0$ as $x\to\infty$ $(y\neq 0, a)$.



* V has no z-dep. $V=V(x,y)$
$= X(x)Y(y)$. (our choice).

$$\therefore \frac{\partial^2}{\partial x^2}(X(x)Y(y)) + \frac{\partial^2}{\partial y^2}(X(x)Y(y)) = 0$$

$$Y(y)\frac{\partial^2 X}{\partial x^2} + X(x)\frac{\partial^2 Y}{\partial y^2} = 0. \Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0.$$

depend on only x — depends on only y

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k^2 \text{ (const)}$$

*** why not $-k^2X$, $+k^2Y$?

$$\frac{d^2X}{dx^2} = k^2 X \ (>0) \Rightarrow X(x) = Ae^{kx} + Be^{-kx}$$

$$\frac{d^2Y}{dy^2} = -k^2 Y \ (<0) \Rightarrow Y(y) = C\sin ky + D\cos ky \quad (\text{or } e^{iky} \text{ or } e^{-iky})$$

$C\sin ky + D\cos ky$: real #; $z_1 e^{iky} + z_2 e^{-iky}$: complex #

$A=0$ due to (iv) $V\to 0$ as $x\to\infty$
$D=0$ due to (i) $V=0$ for $y=0$.

$$\therefore V(x,y) = BCe^{-kx}\sin ky = [\ ]\,e^{-kx}\sin ky.$$

$[\ ]$ → "constant"

$e^{\pm kx} \to +\infty$ or 0 as $x \to \infty$

*** $+k^2 > 0 \Rightarrow$ monotonic funct.

$-k^2 < 0 \Rightarrow$ sinusoidal funct. (oscillation) $\Rightarrow$ bounded solution ($V=0$).

$\Rightarrow$ multiple modes

Fourier series

Now (ii) will set k.

$\sin ka = 0 \to ka = n\pi \quad (n = 1, 2, 3, \cdots)$.

$k = \frac{n\pi}{a}$.

why $n=0$ is excluded? $V(x,y) = 0$

$\Rightarrow$ infinite # of sol'n's running $n = 1, 2, 3, \cdots\cdots$

$\{V_n\}$ for $n = 1, 2, \cdots$

$\therefore$

$V_n = C_n e^{-k_n x} \sin k_n y$

where $k_n = \frac{n\pi}{a}$.

For linear diff. eq's, any linear comb. of the solutions is a sol.

$\nabla^2 V_n = 0 \Rightarrow \nabla^2 V = \alpha_1 \nabla^2 V_1 + \alpha_2 \nabla^2 V_2 + \cdots = 0$.

$\therefore V(x,y) = \sum_{n=1}^{\infty} C_n e^{-k_n x} \sin k_n y = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$

"completeness"

The final b.c. to be used. (iii).

$V(0,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$.

Fourier exp.

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \frac{a}{2}\delta_{nn'}$$

orthogonality.

$$\int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy$$

$$= \int_0^a \sum_{n'=1}^{\infty} C_{n'} \sin\left(\frac{n'\pi y}{a}\right) \sin\left(\frac{n\pi y}{a}\right) dy$$

$$= \sum_{n'=1}^{\infty} \underbrace{\int_0^{a} \sin\left(\frac{n'\pi y}{a}\right) \sin\left(\frac{n\pi y}{a}\right) dy}_{\frac{a}{2}\delta_{nn'}}\, C_{n'} = \frac{a}{2} C_n.$$

$$\therefore C_n = \frac{2}{a} \int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy.$$

If $V_0(y) = V_0$ (const)

$$C_n = \frac{2}{a}V_0\int_0^a \sin\left(\frac{n\pi y}{a}\right) = \frac{2V_0}{a}\left(\frac{a}{n\pi}\right)\left(1-\cos(n\pi)\right)$$

$$= \begin{cases} 0 & \text{for even } n \\ \frac{4V_0}{n\pi} & \text{for odd } n. \end{cases}$$

Finally,

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5,}^{\infty} \frac{1}{n} e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right).$$

* Completeness & Orthogonality.

$\{f_n(y)\}$: a set of fct is complete if any fct $f(y)$ & $\{f_n(y)\}$

$$f(y) = \sum_n C_n f_n(y).$$

$\{f_n(y)\}$ is orthogonal if

$$\int_0^a f_n(y) f_{n'}(y)\, dy = \alpha\, \delta_{n,n'}$$

if $\alpha = 1$, then it is called orthonormal.

EX

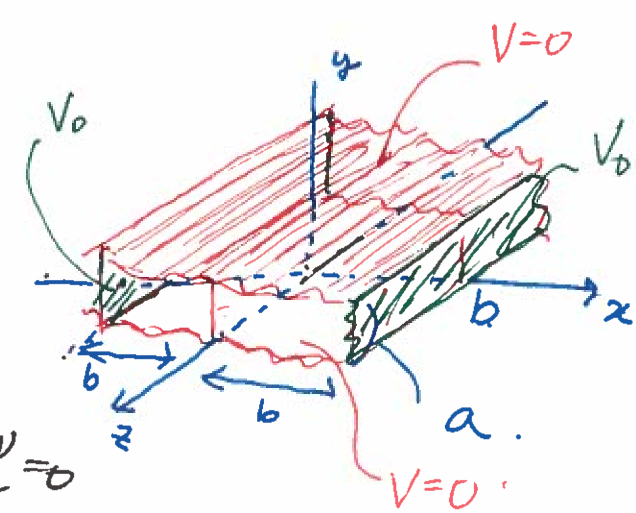
Infinitely long in the z-dir.

(i) $V=0$ for $y=0$

(ii) $V=0$ for $y=a$

(iii) $V=V_0$ for $x=\pm b$.

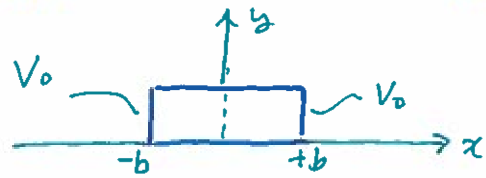
(iv)



Again solving $\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0$

$$V(x,y) = (Ae^{+kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

this term should survive because $V(x\to\infty) \neq 0$.
$V(x = \pm b) = V_0$



Symmetric for $x \to (-x)$.

$$\therefore X(x) = Ae^{+kx} + Be^{-kx} = X(-x)$$

$$\Rightarrow A = B.$$

Now
$$V(x,y) = A(e^{+kx} + e^{-kx})(C\sin ky + D\cos ky)$$
$$= \cosh(kx)(\underline{C}\sin ky + \underline{D}\cos ky)$$

(renormalized to absorb other const.
The bottom line is that we need only 2 const's!)

(i) $V=0$ for $y=0$
$$Y(y=0) = C\sin k0 + D\cos k0 = D = 0.$$

(ii) $V=0$ for $y=a$
$$Y(a) = C\sin ka = 0. \Rightarrow k = \frac{n\pi}{a} \quad (n=1,2,3\cdots).$$

$$\therefore V(x,y) = C\cosh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right).$$

Therefore the general sol'n.

$$V(x,y) = \sum_{n=1}^{\infty} \underline{C_n}\cosh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right).$$

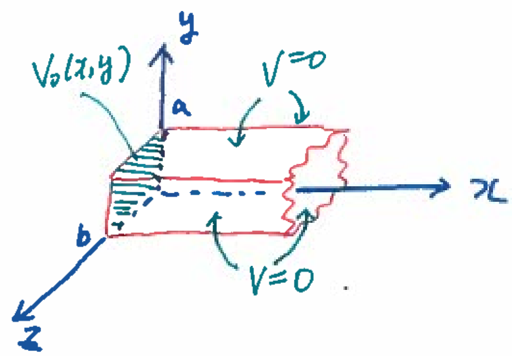
$$V(b,y) = \sum_{n=1}^{\infty} \underbrace{C_n\cosh\left(\frac{n\pi b}{a}\right)}_{\lambda_n}\sin\left(\frac{n\pi y}{a}\right).$$

$$\lambda_n = \begin{cases} 0 & \text{even } n \\ \frac{4V_0}{n\pi} & \text{odd } n. \end{cases}$$

$$\therefore \quad V(x,y) = \frac{4V_0}{n\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} \cdot \frac{\cos(n\pi x/a)}{\cos(n\pi b/a)} \sin(n\pi y/a).$$

Ex 3-D (x, y, z)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$



(i) $V = 0$ for $y = 0$
(ii) $V = 0$ for $y = a$
(iii) $V = 0$ for $z = 0$
(iv) $V = 0$ for $z = b$
(v) $V \to 0$ as $x \to \infty$
(vi) $V = V_0(y, z)$ for $x = 0$.

$$V(x,y,z) = X(x)\,Y(y)\,Z(z).$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

$\frac{1}{X}\frac{d^2X}{dx^2}$	$\frac{1}{Y}\frac{d^2Y}{dy^2}$	$\frac{1}{Z}\frac{d^2Z}{dz^2}$
x only	y only	z only
unbounded.	bounded	bounded
(k^2+l^2)	$-k^2$	$-l^2$

$$\therefore \quad \frac{d^2X}{dx^2} = (k^2+l^2) = m^2 \; ; \quad \frac{d^2Y}{dy^2} = -k^2 Y \; ; \quad \frac{d^2Z}{dz^2} = -l^2 Y$$

$$X(x) = A e^{+mx} + B e^{-mx} \qquad \left(m = \sqrt{k^2+l^2}\right).$$
$$Y(y) = C \sin ky + D \cos ky$$
$$Z(z) = E \sin lz + F \cos lz$$

(v) $V \to 0$ as $x \to \infty$ $\Rightarrow A = 0$.
(i) $V = 0$ for $y = 0$ $\Rightarrow D = 0$
(iii) $V = 0$ for $z = 0$ $\Rightarrow F = 0$

$$\therefore \quad V(x,y,z) = C\, e^{-mx} \sin ky \sin lz$$

(ii) & (iv) $V=0$ for $y=a$ & $z=b$ $\Rightarrow$ $k=\frac{n\pi}{a}$ & $l=\frac{p\pi}{b}$

$(n=1,2,3,\cdots)$ $(p=1,2,3,\cdots)$.

$\therefore$ $m=\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{p\pi}{b}\right)^2}=\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{p}{b}\right)^2}$.

$$V(x,y,z)=\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}C_{n,p}\,e^{-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{p}{b}\right)^2}\,x}\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{p\pi z}{b}\right).$$

(vi)

$$V(0,y,z)=V_0(y,z)=\sum_n\sum_p C_{n,p}\,e^{-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{p}{b}\right)^2}\cdot 0}\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{p\pi z}{b}\right).$$

$$C_{n,p}=\left(\frac{2}{a}\right)\int_0^a \underbrace{V_0(y,z)}_{=V_0\ (\text{const})}\sin\left(\frac{n\pi y}{a}\right)dy\ \left(\frac{2}{b}\right)\int_0^b\sin\left(\frac{p\pi z}{b}\right)dz.$$

$$=\begin{cases}0 & (n \text{ or } p \text{ even})\\ \frac{16V_0}{\pi^2 np} & (n \text{ and } p \text{ odd})\end{cases}.$$

$$\therefore\ V(x,y,z)=\frac{16V_0}{\pi^2}\sum_{n,p=1,3,5}^{\infty}\frac{1}{np}\,e^{-\pi\sqrt{\ \ }\cdot x}\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{p\pi z}{b}\right).$$

HW 3.13, 3.15, 3.16