- Laplace's Eq.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{r}\,')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau' \qquad (1)$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{r}\,')\frac{1}{\mathscr{r}}d\tau' = -\int_\infty^{\vec{r}} \vec{E}(\vec{r}\,')\cdot d\vec{r}\,'$$

In other words, $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$; $\vec{\nabla}\times\vec{E}(\vec{r}) = 0$. ——(2)

Combining with Gauss's Law; $\vec{\nabla}\cdot\vec{E} = \rho(\vec{r})/\varepsilon_0$.

$$\vec{\nabla}\cdot(\vec{\nabla}V(\vec{r})) = \boxed{\nabla^2 V(\vec{r}) = -\rho(\vec{r})/\varepsilon_0} \quad \text{Poisson's Eq.} \qquad (3)$$

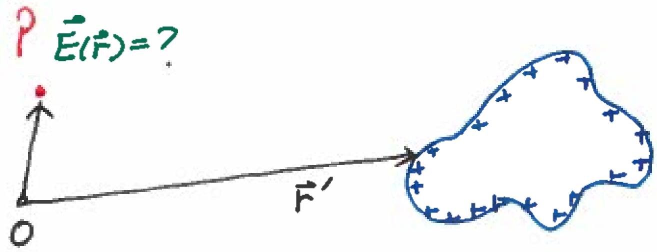
- The ultimate goal in electrostatics is to evaluate $\vec{E}(\vec{r})$ from a given configuration of charges. Eq (1) is the straight expression for this. However calculating integral for an arbitrary $\rho(\vec{r})$ is not an easy task even though ρ is known!
- The same problem can be reformulated with eq (3), in differential form. (You can always do differentiation!)
  - Solving the differential eq. for $V(\vec{r})$ with 2 boundary conditions
  - Then calculate $\vec{E} = -\vec{\nabla}V(\vec{r})$.
- Furthermore, most of the cases we calculate $\vec{E}(\vec{r})$ where no charge is present, i.e. $\rho(\vec{r}) = 0$. Then

$$\boxed{\nabla^2 V(\vec{r}) = 0} \quad \text{Laplace's Eq.}$$

✓ $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2} = 0$$

✓ $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

✓ $$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial\varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$



$\rho(\vec{r}')$ unknown but we know

V on the surface = const $= V_0$
(equipotential surface)

$\Downarrow$

Boundary Condition.

+

Coulomb's Law

$$\vec{F}_1 = -\vec{F}_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathscr{r}_{12}^2}\hat{\mathscr{r}}_{12} \quad \text{where } \vec{\mathscr{r}}_{12} = \vec{r}_1 - \vec{r}_2$$

$$\rightsquigarrow \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{r}')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau' \quad \text{where } \vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

$\Downarrow$

$$\vec{\nabla}\times\vec{E} = 0 \rightarrow \vec{E} = -\vec{\nabla}V \quad \text{(conservative)}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{r}')\frac{1}{\mathscr{r}}\,d\tau'$$

Gauss's Law

$$\vec{\nabla}\cdot\vec{E} = \rho(\vec{r})/\varepsilon_0$$

$$\oint_S \vec{E}\cdot d\vec{a} = \int_V (\vec{\nabla}\cdot\vec{E})\,d\tau$$

$\Rightarrow$

$\Downarrow$

$$\boxed{\nabla^2 V(\vec{r}) = -\rho(\vec{r})/\varepsilon_0 \quad + \text{Boundary Condition}}$$

- Meaning of Laplace's Eq : $\nabla^2 V(\vec{r}) = 0$ in free space w/o $\rho(\vec{r})$

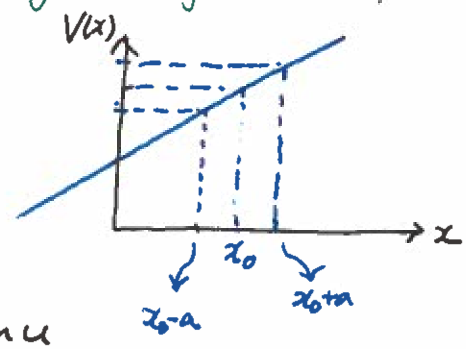
◎ 1-D.

$$\frac{d^2 V(x)}{dx^2} = 0 \Rightarrow V(x) = mx + b$$

Determined by boundary conditions.

(i) $V(x) = \frac{1}{2}\left[V(x+a) + V(x-a) \right]$

The value of the solution at a given point = average values of the solution at two points equal distance apart on both sides.

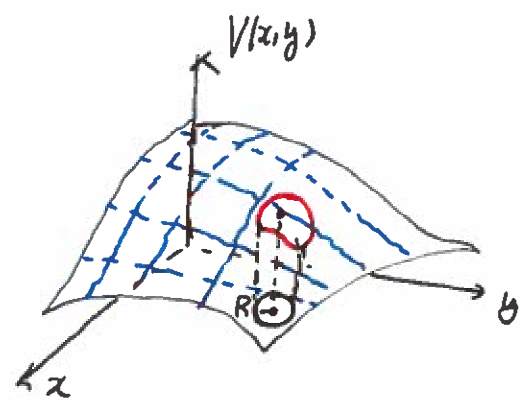


(ii) $\frac{d^2 V}{dx^2} = 0$ for all x $\Rightarrow$ no minimum or maximum other than at the boundary.

◎ 2-D.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

(i) $V(x,y) = \frac{1}{2\pi R} \oint_C V \, dl$.



(ii) $V(x,y)$ has no maxima or minima other than at the boundaries.

However, it is possible $\frac{\partial^2 V}{\partial x^2} > 0$ and $\frac{\partial^2 V}{\partial y^2} < 0$ to satisfy $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

◎ 3-D.

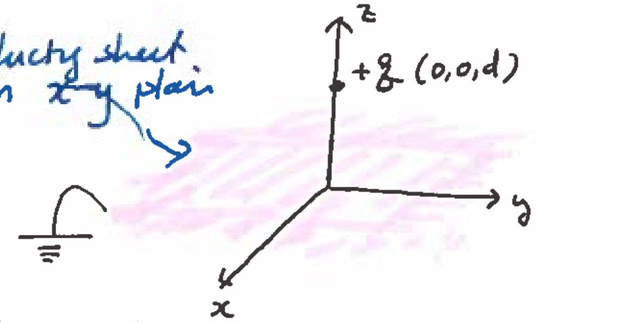
(i) $V(\vec{r}) = \frac{1}{4\pi R^2} \oint_{\text{Sphere}} V \, da$

(ii) $V(\vec{r})$ has no maxima or minima other than at the boundaries.

▸ Solving Laplace's Eq : Technique I (Image Charge Method).

◎ $V(\vec{r})$ for $z>0$? region of interest

conducting sheet on x-y plain



Conducting sheet is grounded.

$\Rightarrow V(z=0) = \boxed{0}$

Boundary Cond.

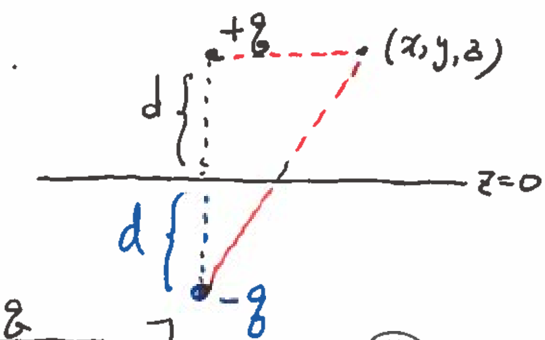
And also $V \to 0$ for $r \to \infty$ (only for $z>0$)

"The idea is to do something in the region of NO interest to produce the correct boundary conditions prescribed in the problem." Then owing to the uniqueness theorem, the ~~resul~~ solution is the solution you are seeking.

What can you do in the region of $z<0$ to produce the boundary condition $V(z=0)=0$ and $V(r\to\infty)=0$.

Put $-q$ at the symmetric point of $+q$.

$-q$ at $(0,0,-d)$.



Then the potential is from the two point charges.

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{+q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right] \quad (*)$$

For $z=0$, $V(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{x^2+y^2}}\right] = 0$

$V(r\to\infty) = 0$.

Therefore, (*) is the unique solution to the problem.

The fake charge placed in the region of No interest is like a mirror image of $+q$.

$\boxed{HW}$ 3.2 .

<u>EX</u> $V(\vec{r}) = V(r)$; only a function of r

$$\nabla^2 V(\vec{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0$$

Determined by B.C.

$$\therefore \; r^2\frac{dV}{dr} = C \;\to\; \frac{dV}{dr} = \frac{C}{r^2} \;\to\; \boxed{V(r) = -\frac{C}{r} + D}$$

$V(\vec{r}) = V(s)$; only a function of s in cylindrical coord.

$$\nabla^2 V(\vec{r}) = \frac{1}{s}\frac{d}{ds}\left(s\frac{dV}{ds}\right) = 0 .$$

$$s\frac{dV}{ds} = C \;\to\; \frac{dV}{ds} = \frac{C}{s} \;\to\; \boxed{V(s) = C\ln s + D .}$$

- Uniqueness Theorem

$\vec{\nabla}\cdot\vec{v}(\vec{r}) = f(\vec{r})$ & $\vec{\nabla}\times\vec{v}(\vec{r}) = g(\vec{r})$.

"A vector field whose curl and divergence are known everywhere is uniquely determined, provided the sources vanish at infinity and also the field is also required to vanish at infinity, at least as rapidly as $1/r^2$"

⇒ "The solution to Laplace's eq in some volume is uniquely determined if V is specified on the boundary surface."

⇒ In a volume surrounded by conductors and containing specified charge density ρ, the electric field is uniquely ~~defined~~ determined if the total charge on each conductor is given

[i] The solution to Laplace's eq in some vol. V is uniquely determined if V is specified on the boundary surface S

[ii] In a volume V surrounded by conductors and containing a specific charge density, $\vec{E}$ is uniquely determined if the total charge on the conductor is given.

- $+q$ will induce surface charge on the conducting sheet drawing negative charges from the ground reservoir. Then what is the induced surface charge σ?

always from the boundary condition.

$$\delta(\hat{n}\cdot\vec{\nabla}V(\vec{r})) = -\frac{\sigma}{\varepsilon_0}$$

$$\hat{n}\cdot\vec{\nabla}V\Big|_{above} - \hat{n}\cdot\vec{\nabla}V\Big|_{below} = -\frac{\sigma}{\varepsilon_0}.$$

Induced charge on the surface of a conductor

$$\sigma = -\varepsilon_0\frac{\partial V}{\partial n} \qquad \text{since } V_{below}=0.$$

$$= -\varepsilon_0\frac{\partial V}{\partial z}\Big|_{z=0}$$

$$= -\varepsilon_0\frac{1}{4\pi\varepsilon_0}\left\{\frac{-q(z-d)}{\{x^2+y^2+(z-d)^2\}^{3/2}} + \frac{q(z+d)}{\{x^2+y^2+(z+d)^2\}^{3/2}}\right\}\Bigg|_{z=0}$$

$$\sigma(x,y,0) = \frac{-2qd}{2\pi(x^2+y^2+d^2)^{3/2}}.$$

surface charge with this given distribution at $z=0$
$\equiv$ $-q$ point charge at $(0,0,-d)$.

Obviously, then

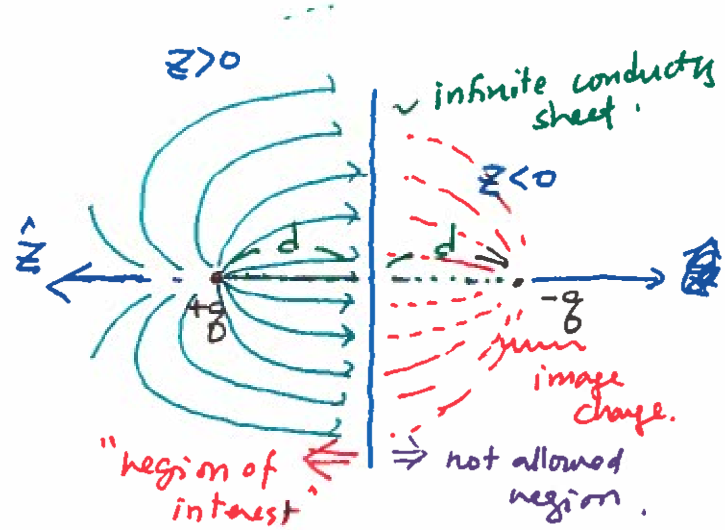
$$\iint_{x\text{-}y\text{ plain}} \sigma\, da = -q.$$

- Force & Energy

$$\vec{F} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{z}$$

$$W \neq qV = \frac{1}{4\pi\varepsilon_0}\frac{q(-q)}{(2d)}$$

this includes the energy in the region ($z<0$) Not Allowed!

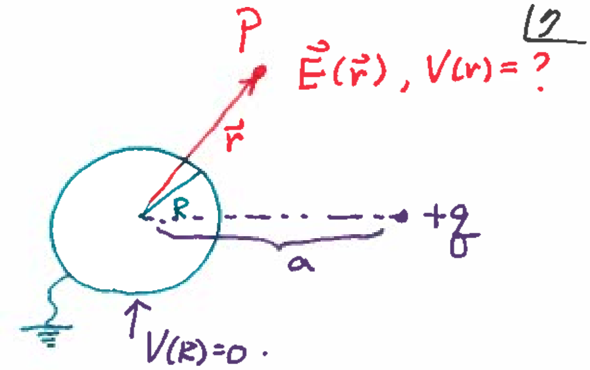


$$W = \frac{\varepsilon_0}{2}\int E^2\, d\tau \quad (\text{integral for } z>0).$$

$$= -\frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{q^2}{(2d)}.$$

EX

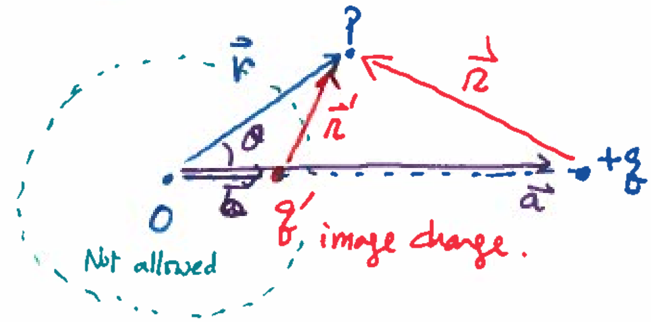
$$\begin{cases} q' = -\frac{R}{a} q \\ b = \frac{R^2}{a} \end{cases}$$



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right)$$

$$\vec{r} = \vec{b} + \vec{\mathscr{r}}' \qquad \vec{\mathscr{r}}' = \vec{r} - \vec{b}$$

$$\vec{r} = \vec{a} + \vec{\mathscr{r}} \qquad \vec{\mathscr{r}} = \vec{r} - \vec{a}$$



$$\therefore (\mathscr{r}')^2 = (\vec{r}-\vec{b})\cdot(\vec{r}-\vec{b})$$
$$= r^2 + b^2 - 2\vec{r}\cdot\vec{b}$$
$$= r^2 + b^2 - 2rb\cos\theta = r^2 + \left(\frac{R^2}{a}\right)^2 - 2r\frac{R^2}{a}\cos\theta.$$

$$(\mathscr{r}^2) = r^2 + a^2 - 2ra\cos\theta$$

$$\therefore V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} + \frac{\left(-\frac{R}{a}\right)q}{\sqrt{r^2+\left(\frac{R^2}{a}\right)^2-2r\frac{R^2}{a}\cos\theta}}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{1}{\sqrt{R^2+\left(\frac{ra}{R}\right)^2-2ra\cos\theta}}\right]$$

$$V(r=R) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{R^2+a^2-2Ra\cos\theta}} - \frac{1}{\sqrt{R^2+a^2-2Ra\cos\theta}}\right] = 0.$$

correct b.c.

therefore, $V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right)$.

Force, $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(a-b)^2}\hat{r} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2Ra}{(a^2-b^2)^2} \propto \frac{Q^2}{D^2}$.

$\boxed{\text{HW}}$ 3.7. & 3.8

(b) $\sigma = -\varepsilon_0\frac{\partial V}{\partial n} = -\varepsilon_0\frac{\partial V}{\partial r}$