L1 Ch. 1 Vector Analysis.

▶ Notations & Conventions.

- Vectors : $\vec{A}$, $\underset{\sim}{A}$; unit vectors $\hat{e}$, $\hat{r}$

  Matrices : $\mathbb{R}$, $\mathbb{T}$

- In Cartesean Coordinates, the basis vectors $(\hat{x}, \hat{y}, \hat{z})$; $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$; $(\hat{e}_x, \hat{e}_y, \hat{e}_z)$.

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} = A_x \hat{e}_1 + A_y \hat{e}_2 + A_z \hat{e}_3 = A_x \hat{e}_x + A_y \hat{e}_y + A_z \hat{e}_z$$
$$= \sum_{i=1}^{3} A_i \hat{e}_i$$

Indices $(x, y, z) \leftrightarrow (1, 2, 3)$ interchangeable!

$$\vec{B} = \sum_{i=1}^{3} B_i \hat{e}_i \qquad (i : \text{summation index})$$

$$\vec{A} \cdot \vec{B} = \left(\sum_{i=1}^{3} A_i \hat{e}_i\right) \cdot \left(\sum_{j=1}^{3} B_j \hat{e}_j\right).$$
$$= \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \, \hat{e}_i \cdot \hat{e}_j \quad : \text{Scalar quantity.} \qquad \text{—— (1)}$$

$$\vec{A} \times \vec{B} = \left(\sum_{i=1}^{3} A_i \hat{e}_i\right) \times \left(\sum_{j=1}^{3} B_j \hat{e}_j\right)$$
$$= \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \, (\hat{e}_i \times \hat{e}_j). \qquad \text{—— (2)}$$

- Kronecker-delta and Levi-Civita symbols.

  - Kronecker-delta (related to • product).

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

therefore we can write

$$\boxed{\hat{e}_i \cdot \hat{e}_j = \delta_{ij}}$$

Eq. ① can be expressed

$$\vec{A}\cdot\vec{B} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_i B_j \delta_{ij}$$

$$= A_i B_j \delta_{ij} = A_i B_i$$

* whenever you have an index (e.g. i or j) appearing more than once in an expression, the summation over the index is implied.: "Einstein Summation Convention"

- Non-repeated index should not be summed over.
- Your final result after performing appropriate summation should only have singly appearing index.

EX

$$A_i B_i C_j = C_j \sum_{i=1}^{3} A_i B_i = C_j (A_1B_1 + A_2B_2 + A_3B_3)$$

$$A_i B_i C_j D_j = \sum_{i=1}^{3} A_i B_i \sum_{j=1}^{3} C_j D_j = (A_1B_1 + A_2B_2 + A_3B_3)(C_1D_1 + C_2D_2 + C_3D_3)$$

- Levi-Civita (related to X-product)

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for even permutations of } (1\,2\,3) \text{ with } i\neq j\neq k \\ -1 & \text{for odd permutations of } (1\,2\,3) \text{ with } i\neq j \neq k. \end{cases}$$

$$= 0 \quad \text{otherwise}.$$

$$\epsilon_{iik} = \epsilon_{iii} = \epsilon_{ijj} = \cdots = 0 .$$

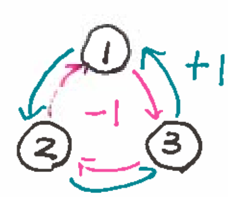
$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji} \quad \text{(odd permutation)}$$

$$= \epsilon_{jki} = \epsilon_{kij}$$

Set $\epsilon_{123} = +1 \Rightarrow \epsilon_{321} = -1$

$= \epsilon_{231} \qquad = \epsilon_{132}$

$= \epsilon_{312} \qquad = \epsilon_{213}$



Useful identities

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$ ← (i) does not appear!

$$\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km}$$ ← (i) (j) do not appear.

$$\epsilon_{ijk}\epsilon_{ijk} = 3! = 6.$$

$\boxed{\text{HW}}$: Verify the above relations.

$$\vec{C} = \vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \quad \text{— (3)}$$

$$= (A_2B_3 - A_3B_2)\hat{e}_1 + (A_3B_1 - A_1B_3)\hat{e}_2 + (A_1B_2 - A_2B_1)\hat{e}_3$$

$$= \epsilon_{ijk}A_jB_k\hat{e}_i$$

$$= \epsilon_{1jk}A_jB_k\hat{e}_1 + \epsilon_{2jk}A_jB_k\hat{e}_2 + \epsilon_{3jk}A_jB_k\hat{e}_3$$

$$\underbrace{\epsilon_{123}}_{+1}A_2B_3 + \underbrace{\epsilon_{132}}_{-1}A_3B_2$$

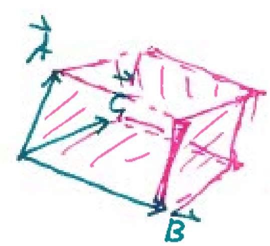
Therefore Eq. (3)

$$C_i\hat{e}_i = \epsilon_{ijk}A_jB_k\hat{e}_i = -\epsilon_{ikj}B_kA_j\hat{e}_i = -\vec{B}\times\vec{A}$$

EX. Show that

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}).$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = A_i(B\times C)_i$$
$$= A_i\epsilon_{ijk}B_jC_k = \epsilon_{ijk}A_iB_jC_k$$
$$= B_j\epsilon_{ijk}C_kA_i = B_j\epsilon_{jki}C_kA_i$$
$$= B_j(\vec{C}\times\vec{A})_j = \vec{B}\cdot(\vec{C}\times\vec{A}).$$



* Geometrical meaning of $\vec{A}\cdot(\vec{B}\times\vec{C})$ is $|\vec{A}\cdot(\vec{B}\times\vec{C})|$ = volume of the parallepiped formed by $\vec{A}, \vec{B}, \& \vec{C}$.

EX BAC-CAB Rule: $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$.

$$\vec{A}\times(\vec{B}\times\vec{C}) = \epsilon_{ijk} A_j (\vec{B}\times\vec{C})_k \hat{e}_i$$

$$= \epsilon_{ijk} A_j \epsilon_{klm} B_l C_m \hat{e}_i$$

$$= \epsilon_{ijk}\epsilon_{klm} A_j B_l C_m \hat{e}_i$$

$\epsilon_{ijk}\epsilon_{klm} = \epsilon_{kij}\epsilon_{klm}$
$= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j B_l C_m \hat{e}_i$$

$$= (A_j B_i C_j - A_j B_j C_i)\hat{e}_i$$

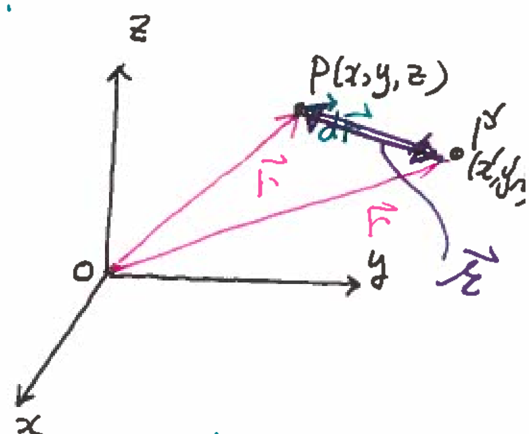
$$= (B_i\hat{e}_i)(A_j C_j) - (C_i\hat{e}_i)(A_j B_j)$$

$$= \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}).$$

HW Prob 1.6

- Displacement vector & infinitesimal displacement vector

a vector from the origin to a point in space (x, y, z).
∴ it depends on the coordinate system.



$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$r = |\vec{r}| = (x^2+y^2+z^2)^{1/2}$$

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}}$$

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

infinitesimal displacement vector (integral, differential element)

$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$ (from P' to P)

$$= (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}.$$

- Additional vector identities:

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$$ show this!

$$\vec{A}\times[\vec{B}\times(\vec{C}\times\vec{D})] = \vec{B}[\vec{A}\cdot(\vec{C}\times\vec{D})] - (\vec{A}\cdot\vec{B})(\vec{C}\times\vec{D})$$

$\boxed{\text{HW}}$

(i) What is the value of δ_{ii}?

(ii) $\mathbb{A}$ is a 3×3 matrix with elements A_{ij}. Show that

$$\det \mathbb{A} = \frac{1}{3!}\,\epsilon_{ijk}\,\epsilon_{q\ell m}\,A_{iq}A_{j\ell}A_{km}$$