L36 Constraint Forces & Lagrange multipliers

Lagrange formalism doesnot require explicitely define constraint forces, BUT

a) a set of independent generalized coordinates $q_1, \dots q_N$ may not be readily identified.

b) sometimes we may need to know constraint forces F_{cons}

* 2D problem (example) in xy plane. only one degree of freedom: like pendulum x & y are related to each other by the constraint equation $f(x,y) = \text{const}$.

for a pendulum $f(x,y) = x^2+y^2 = \ell^2 = \text{const}$

for Atwood machine $f(x,y) = x+y = \ell = \text{const}$

* Hamiltonian principle $S = \int_{t_1}^{t_2} \mathcal{L}(x,\dot{x},y,\dot{y})dt$

$x(t)$ & $y(t)$ are such that S is stationary.

right path → wrong path

$x(t) \longrightarrow x(t) + \delta_x(t)$

$y(t) \longrightarrow y(t) + \delta_y(t)$

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial x}\delta_x + \frac{\partial \mathcal{L}}{\partial \dot{x}}\dot{\delta}_x + \frac{\partial \mathcal{L}}{\partial y}\delta_y + \frac{\partial \mathcal{L}}{\partial \dot{y}}\dot{\delta}_y\right) dt = 0$$

use $\underbrace{\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\delta_x\right)}_{0} = \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\delta_x + \int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot{x}}\dot{\delta}_x$

because $\delta_x(t_1) = \delta_x(t_2) = 0$

$$\delta S = \int_{t_1}^{t_2}\left(\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\delta_x\, dt + \int_{t_1}^{t_2}\left(\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)\delta_y\, dt$$

$\delta S = 0$ for 2 DoF with x, y to be 2 independent generalized coordinates.

if x & y are dependent $\delta S = 0$ only for δ_x & δ_y consistent with the constraint $f(x,y) = \text{const}$ or

$$\delta f = \frac{\partial f}{\partial x}\delta_x + \frac{\partial f}{\partial y}\delta_y = 0$$

L36 adding $\lambda \cdot \delta f$ to δS does not change the resut or.

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial x} + \lambda \frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right]\delta_x dt + \int \left[\frac{\partial \mathcal{L}}{\partial y} + \lambda \frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}\right]\delta_y dt$$

$\lambda(t)$ – Lagrange multiplier

by selecting $\lambda(t)$ we can always have

$$\frac{\partial \mathcal{L}}{\partial x} + \frac{\partial f}{\partial x}\lambda(t) - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = 0 \quad + \text{ y equation.}$$

constraint equation $f(x,y) = \text{const}$

3 unknow functions $x(t), y(t), \lambda(t) \Leftrightarrow 3$ dif. eq.

* Constrained force. $\mathcal{L} = T(\dot{x},\dot{y}) - U(x,y)$

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt}P_x = \dot{P}_x$$

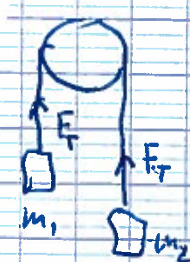
$$F_x + F_{xc} = \dot{P}_x \qquad F_y + F_{yc} = \dot{P}_y$$

$$F_{xc} = \lambda \frac{\partial f}{\partial x} \qquad F_{yc} = \lambda \frac{\partial f}{\partial y}$$

* Atwood machine : $\mathcal{L} = T - U =$

$$\frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + m_1 g x + m_2 g y$$

$$f(x,y) = x + y = \text{const.}$$



$$\frac{\partial \mathcal{L}}{\partial x} + \lambda \frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \rightarrow m_1 g + \lambda = m_1 \ddot{x}$$

$$\frac{\partial \mathcal{L}}{\partial y} + \lambda \frac{\partial f}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \rightarrow m_2 g + \lambda = m_2 \ddot{y}$$

$$\ddot{x} = -\ddot{y} \qquad \lambda = -F_T$$

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