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EL equation for unconstrained motion
particle moving in 3D $T = \frac{1}{2} m \dot{\vec{r}}^2$
potential energy $U = U(\vec{r}) = U(x, y, z)$

The Lagrangian is: $\mathcal{L} = T - U$

$$\mathcal{L} = \mathcal{L}(x, \dot{x}, y, \dot{y}, z, \dot{z}) \quad x = x(t) \quad y = y(t) \quad z = z(t)$$

$$\frac{\partial \mathcal{L}}{\partial x} = - \frac{\partial U}{\partial x} = F_x \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m \dot{x} = p_x$$

$$N. \text{ second Law } F_x = \dot{p}_x = \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} =$$

N second Law is equivalent to EL equation.

* Hamilton principle

$$\text{action integral } A = \int_{t_1}^{t_2} \mathcal{L} dt$$

the path which a particle follows
between points 1 & 2 in a given time
interval t_1 to t_2 is such that
the action integral is stationary
along that path.

$$\vec{F} = m\vec{a} \equiv \frac{\partial \mathcal{L}}{\partial x_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \equiv x, y, z(t) \text{ determined by stat. action integral.}$$

* generalized coordinates

$$q_i = q_i(\vec{r}) \quad i=1,2,3$$

$$\vec{r} = \vec{r}(q_1, q_2, q_3)$$

The Lagrangian is $\mathcal{L}(q_1, \dot{q}_1, q_2, \dot{q}_2, q_3, \dot{q}_3)$

$$A = \int_{t_1}^{t_2} \mathcal{L}(q_i, \dot{q}_i) dt \rightarrow \frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

* Particle in 2D

case (a) Cartesian coordinates.

$$\mathcal{L} = T - U = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - U(x, y)$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \rightarrow F_x = m \ddot{x}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} \rightarrow F_y = m \ddot{y}$$

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Case (b) Polar coordinates.

$$T - U = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) - U(r)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m r \dot{\varphi}^2 - \frac{\partial U}{\partial r} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \right) = \frac{d}{dt} (m r^2 \dot{\varphi}) = m \dot{r}^2$$

$$- \frac{\partial U}{\partial r} = F_r = m(\ddot{r} - r\dot{\varphi}^2)$$

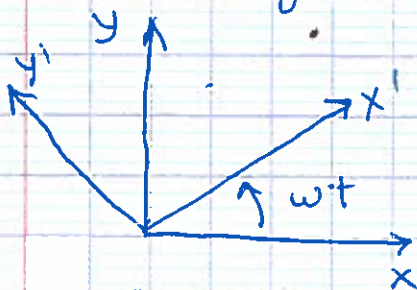
$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \rightarrow - \frac{\partial U}{\partial \varphi} = \frac{d}{dt} (m r^2 \dot{\varphi})$$

$$\nabla_{\vec{r}} U = \frac{\partial U}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial U}{\partial \varphi} \hat{\varphi} \quad \vec{F}_{\varphi} = - \frac{1}{r} \frac{\partial U}{\partial \varphi}$$

$$r \vec{F}_{\varphi} = \Gamma - \text{torque} \rightarrow F_{\varphi} r = \Gamma = \frac{d}{dt} (m r^2 \dot{\varphi})$$

$$\text{or } \Gamma = \frac{dL}{dt} \quad L = m r^2 \dot{\varphi} - \text{angular momentum}$$

* Rotating coordinates.



$$T = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) \quad U(x, y) = 0$$

$$x = x' \cos \omega t + y' \sin \omega t$$

$$y = y' \cos \omega t - x' \sin \omega t$$

$$\dot{x} = \dot{x}' \cos \omega t + \dot{y}' \sin \omega t - \omega x' \sin \omega t + \omega y' \cos \omega t =$$

$$= (\dot{x}' + y' \omega) \cos \omega t + (\dot{y}' - x' \omega) \sin \omega t$$

$$\dot{y} = (\dot{y}' - x' \omega) \cos \omega t - (\dot{x}' + y' \omega) \sin \omega t$$

$$\dot{x}^2 + \dot{y}^2 = (\dot{x}' + y' \omega)^2 + (\dot{y}' - x' \omega)^2$$

$$\mathcal{L} = \frac{1}{2} m (\dot{x}'^2 + \dot{y}'^2) + \frac{1}{2} m \omega^2 (y'^2 + x'^2) + m \omega (x' y' - y' x') + \text{Coriolis term}$$

$$U_c = -\frac{1}{2} m \omega^2 (x'^2 + y'^2) - \text{"potential energy" due to centrifugal force}$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}'} = \frac{d}{dt} (m \dot{x}' + m \omega y') = m \omega^2 x' - m \omega y'$$

$$m \ddot{x}' = m \omega^2 x' - 2m \omega y'$$