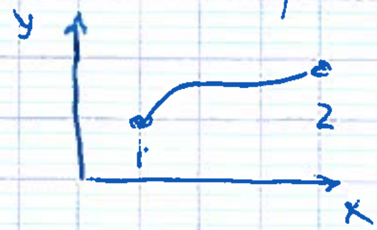
L29

Applications of Euler-Lagrange eq.

* Shortest path between 2 pts on a plane.



$ds = \sqrt{dx^2+dy^2} = dx\sqrt{1+y'^2}$

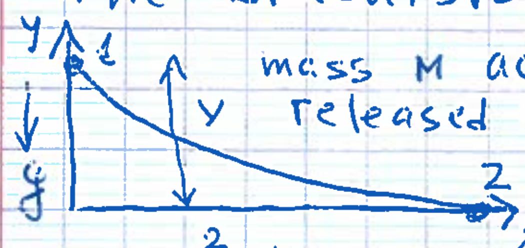
$f(y, y', x) = \left[1+y'^2\right]^{1/2}$

use $\frac{\partial f}{\partial y} = \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]$; $\cancel{0} = \frac{d}{dx}\left[\underbrace{\frac{y'}{\sqrt{1+y'^2}}}_{=\text{const.}}\right]$

$y'^2 = c\,(1+y'^2) \to y'^2 = \text{const} \to y' = a$

$y = ax + b$ — equation for stright line.

* The brachistochrone



mass M accelerated by gravity released at point 1. for what path y(x) it will arrive to point 2 at shortest time

$T_{12} = \int_1^2 \frac{ds}{v}$ $\quad v = \sqrt{2gy}$ $\quad ds = \sqrt{x'(y)^2+1}\,dy$

$T_{12} = \frac{1}{\sqrt{2g}}\int_1^2 \frac{1}{\sqrt{y}}\sqrt{x'(y)^2+1}\,dy$ $\quad f(x, x', y) = \frac{1}{\sqrt{y}}\sqrt{1+x'(y)^2}$

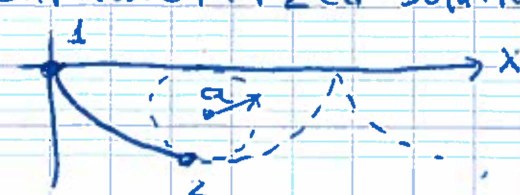
use $\frac{\partial f}{\partial x} = \frac{d}{dy}\frac{\partial f}{\partial x'}$; $\frac{\partial f}{\partial x} = 0$ $\quad \frac{\partial f}{\partial x'} = \frac{1}{\sqrt{y}}\frac{x'}{\sqrt{1+x'^2}} = \text{const.}$

or $\frac{x'^2}{y(1+x'^2)} = \frac{1}{2a}$ $\quad x' = \frac{\sqrt{y}}{\sqrt{2a-y}}$, $\quad x(y) = \int\frac{\sqrt{y}\,dy}{\sqrt{2a-y}}$

use $\boxed{y = a(1-\cos\theta)}$ $\quad x = \int\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{1/2} a\sin\theta\, d\theta \to$

$= a\int(1-\cos\theta)\,d\theta = a(\theta - \sin\theta) + \text{const}$

$\boxed{x = a(\theta - \sin\theta)}$ instead solution $y(x)$ we find parametrized solution $y(\theta)$ and $x(\theta)$



oscillations of a mass on a cycloid track are isochronous

L29 More variables. $x = x(t)$ $y = y(t)$

$t = u$ in the textbook.

$dx = \frac{dx}{du}du$ $dy = \frac{dy}{du}du = y'du$

$ds = du\sqrt{x'^2 + y'^2}$

$L = \int_1^2 du\sqrt{x'^2+y'^2} = \int_1^2 f(x,y,x',y',u)\,du$

the same like for one function

$x = x_0(u) + \alpha\,\xi(u)$ $y = y_0(u) + \beta\,\eta(u)$

$\xi(u_1) = \xi(u_2) = 0$ $\eta(u_1) = \eta(u_2) = 0$

$\frac{\partial S}{\partial \alpha} = 0$ $\frac{\partial S}{\partial \beta} = 0$ $\Longrightarrow$

$\frac{\partial f}{\partial x} = \frac{d}{du}\frac{\partial f}{\partial x'}$ & $\frac{\partial f}{\partial y} = \frac{\partial}{\partial u}\frac{\partial f}{\partial y'}$

* Shortest path again.

$f(x,x'\,y,y',u) = \sqrt{x'^2+y'^2}$

$\frac{\partial f}{\partial x} = 0$; $\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{x'^2+y'^2}}$; $\frac{\partial f}{\partial y} = 0$; $\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{x'^2+y'^2}}$

$\frac{\partial f}{\partial x'} = \text{const}$, $\frac{\partial f}{\partial y'} = \text{const}_2$; ratio $= \frac{y'}{x'} = \frac{dy}{dx} = a$

$y(x) = ax + b$

* a) $f = f(y', x)$ first integral of EL eq

$\hookrightarrow \frac{\partial f}{\partial y'} = \text{const}$

b) $f() = f(y, y')$ does not depend on x

$\Rightarrow \frac{df}{dx} = \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''$ but $\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$

$\Rightarrow \frac{df}{dx} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)y' + \frac{\partial f}{\partial y'}y'' = \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)$