L24

Fourier Series

consider a periodic function $f(t+\tau)=f(t)$

so τ is the period

such periodic function can be presented as a superposition of harmonic function

$\sin(n\pi t/\tau)$ & $\cos(n\pi t/\tau)$ $\omega = 2\pi/\tau$

$\sin(n\omega t)$ & $\cos(n\omega t)$

$$f(t) = \sum_{n=0}^{n=\infty} a_n \cos(\omega t n) + b_n \sin(\omega n t)$$

How to find a_n & b_n ?

$$a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos(n\omega t)\,dt =$$

$$= \frac{2}{\tau}\Big[\sum_m a_m \int \cos(m\omega t)\cos(n\omega t)\,dt + \sum b_m \int \sin(m\omega t)\cos(n\omega t)$$

need to evaluate integrals

$$\left.\begin{aligned} &\int_{-\tau/2}^{\tau/2} \cos(m\omega t)\cos(n\omega t)\,dt \\ &\int_{-\tau/2}^{\tau/2} \sin(m\omega t)\sin(n\omega t)\,dt \\ &\int_{-\tau/2}^{\tau/2} \sin(m\omega t)\cos(n\omega t)\,dt \end{aligned}\right\}$$

defined by orthogonality conditions. of sin & cos

use: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\cos\alpha\cos\beta = \dfrac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$

$*$ $$\int_{-\tau/2}^{\tau/2} \cos(m\omega t)\cos(n\omega t)\,dt = \frac{1}{2}\int \cos((n+m)\omega t)\,dt + \frac{1}{2}\int \cos((n-m)\omega t)$$

case $n=m$: $\frac{1}{2}\int_{-\tau/2}^{\tau/2} dt = \frac{\tau}{2}$

case $n\neq m$: $\frac{1}{2}\int \cos((n\pm m)\omega t)\,dt = \frac{1}{2}\dfrac{\sin((n\pm m)\omega t)}{(n\pm m)\omega}$

$\Big|^{\tau/2}$ $\sin((n\pm m)\omega\,\tau/2) = \sin k\pi = 0$

$*$ $$\int_{-\tau/2}^{\tau/2} \sin(m\omega t)\sin(n\omega t)\,dt = \frac{1}{2}\int \cos((n-m)\omega t)\,dt - \frac{1}{2}\int \cos((n+m)\omega t)\,dt$$

again: $n=m \to \tau/2$

$n\neq m \to 0$

L24

x-term $\int_{-T/2}^{T/2} \sin(m\omega t)\cos(n\omega t)\,dt$.

use: $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$

prove that $\int_{-T/2}^{T/2}\left[\sin(n\pm m)\omega t \mp \sin(n-m)\omega t\right]dt = 0$

* $a_0 = \int f(t)\,dt$.

* FS solutions.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) = D(x)$$

D is a linear operator: for $x = x_1 + x_2$

$$D(x) = D(x_1) + D(x_2) = f_1 + f_2 = f$$

$$f(t) = \sum_n f_n(t) = \sum_n (a_n\cos\omega_n t + b_n\sin\omega_n t)$$

$$x(t) = \sum_n x_n(t)$$ $x_n(t)$ is a solution of $\ddot{x}_n + 2\beta\dot{x}_n + \omega_0^2 x_n = f_n$

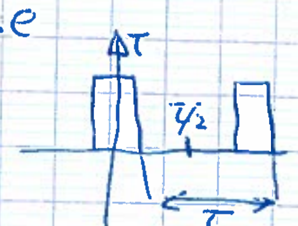
* for $f(t) = \sum f_n\cos(n\omega t)$

$$x_n(t) = A_n\cos(n\omega t - \delta_n)$$

$$A_n^2 = \frac{f_n^2}{(\omega_0^2 - n^2\omega^2)^2 + 4\beta^2 n^2\omega^2} \qquad \delta_n = \arctan\left[\frac{2\beta n}{\omega_0^2 - n^2\omega^2}\right]$$

$$x(t) = \sum_{n=0}^{\infty} A_n\cos(n\omega t - \delta_n)$$

* FS of rectangular pulse



$$a_n = \frac{2}{\tau}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt =$$

$$= \frac{2f_0}{\tau}\int_{-\Delta T/2}^{\Delta T/2}\cos(n\omega t)\,dt =$$

$$= \frac{4f_0}{\tau}\int_0^{\Delta\tau/2}\cos(n\omega t)\,dt = \frac{2f_0}{n\pi}\sin\frac{\pi n\Delta\tau}{\tau}$$

$$f(t) = a_0 + \sum_n a_n\cos(\omega t n) \qquad (b_n = 0)$$

prove.