L22 Driven Dumped Oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F(t)/m = f(t)$$

$F(t)$ - driving force

- Linear differential operators

$$D = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2 \qquad D(x) = f(t)$$

Linear: $D(\alpha x) = \alpha D(x)$; $D(x_1 + x_2) = D(x_1) + D(x_2)$

$D(x) = 0$ - homogeneous equation

$D(x) = f$ - inhomogeneous equation

Particular solution: $D(x_p) = f$

Homogeneous solution: $D(x_H) = 0$

$D(x_p + x_H) = f$ $\quad x_p + x_h$ - also a solution.

- $f(t) = f_0 \cos(\omega t)$ - harmonic driving force

ω - driving frequency

ω_0 - natural frequency

$$\left.\begin{aligned} \ddot{x} + 2\beta\dot{x} + \omega_0^2 x &= f_0 \cos\omega t \\ \ddot{y} + 2\beta\dot{y} + \omega_0^2 y &= f_0 \sin\omega t \end{aligned}\right| \rightarrow \ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$$

$z = x + iy$

look for a solution in a form $\quad z(t) = Ce^{i\omega t}$

get axilliary equation: $(-\omega^2 + 2i\beta\omega + \omega_0^2)C = f_0$

$$C = f_0/(\omega_0^2 - \omega^2 + 2i\beta\omega) = Ae^{-i\delta}$$

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

$$tg(\delta) = 2\beta\omega/(\omega_0^2 - \omega^2)$$

$$z(t) = Ae^{i(\omega t - \delta)} \rightarrow x(t) = \mathrm{Re}\, z(t) = A\cos(\omega t - \delta)$$

general solution: $x(t) = \overbrace{A\cos(\omega t - \delta)}^{x_p(t)} + \underbrace{\overbrace{C_1 e^{r_1 t} + C_2 e^{r_2 t}}^{x_H(t)}}_{\text{transient, which dies out}}$

* $x_H(t)$ depends on initial conditions, but
* it dies out at large t
* $x_p(t)$ - repeats the driving force with a phase shift δ