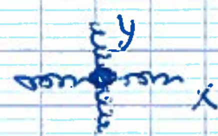
L21 **2D oscillators** $\vec{F} = -k\vec{r}$

$F_x = -kx \quad F_y = -ky$ — central force



for small x & y displacement

$\ddot{x} = -\frac{k}{m}x \quad \ddot{y} = -\frac{k}{m}y \quad \omega^2 = \frac{k}{m}$

* Consider phase-shifted cos solution

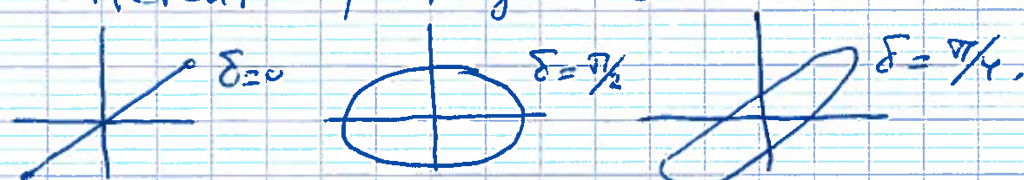
$x(t) = A_x\cos(\omega t - \delta_x)$ — define new time $\omega t' = \omega t - \delta_x$

$y(t) = A_y\cos(\omega t - \delta_y)$

$\rightarrow x(t) = A_x\cos(\omega t')$

$\rightarrow y(t) = A_y\cos(\omega t' - (\delta_y - \delta_x)) = A_y\cos(\omega t' - \delta)$

x-y oscillator trajectory could be quite different depending on δ:



* Anisotropic oscillator: $F_x = -k_x x$, $F_y = -k_y y$

$x(t) = A_x\cos(\omega_x t)$

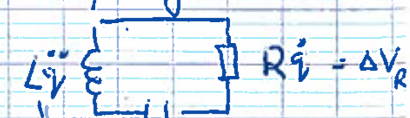
$y(t) = A_y\cos(\omega_y t - \delta)$

- ω_x/ω_y = rational → Lissajous figures
- ω_x/ω_y = irrational → quasiperiodic motion

* Damped oscillations: example damping force $-b\vec{v}$

$m\ddot{x} + b\dot{x} + kx = 0$

$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0$



homogeneous equation $\quad \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \qquad 2\beta = b/m \quad \omega_0^2 = \frac{k}{m}$

β – damping constant

ω_0 – natural frequency

Lets find solution in the form: $x(t) = e^{rt}$

$(r^2 + 2\beta r + \omega_0^2)e^{rt} = 0$ – auxiliary equation

$r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2} \qquad r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$

General solution: $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

221

a) $\beta = 0$ no dumping

$x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$

b) $\beta^2 < \omega_0^2$ – weak dumping

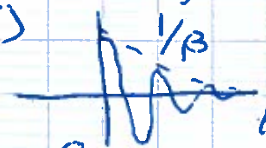
$\sqrt{\beta^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \beta^2} = i\omega_1$

eventually decay $e^{-\beta t}$

$x(t) = e^{-\beta t}(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t})$

$= A e^{-\beta t}\cos(\omega_1 t - \delta)$

$1/\beta$ – dumping time



c) $\beta > \omega_0$ – strong dumping (over dumping)

$x(t) = e^{-\beta t}(C_1 e^{-\alpha t} + C_2 e^{+\alpha t})$ $\alpha = \sqrt{\beta^2 - \omega_0^2}$

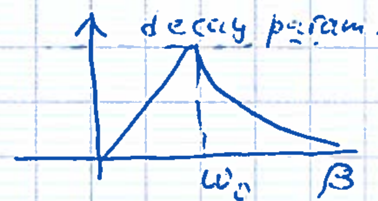
↑dominant term

d) Critical dumping $\beta = \omega_0$ $r_1 = r_2$

need one more solution: $x_2 = t e^{-\beta t}$

$x(t) = e^{-\beta t}(C_1 + tC_2)$

used for shock absorbers



* Driven Dumped oscillators

$\ddot{x} + 2\beta\dot{x} + \omega_0 x = F(t)/m = f(t)$

$F(t)$ – driving force

* Linear differential operators

$D = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$ for equation above

Linear: $D(\alpha x) = \alpha D(x)$ $D(x_1 + x_2) = D(x_1) + D(x_2)$

$Dx = 0$ – homogeneous equation

$Dx = f$ – inhomogeneous equation

– particular solution $Dx_p = f$ $D(x_p + \alpha x_h) = f$

– homogeneous solution $Dx_h = 0$