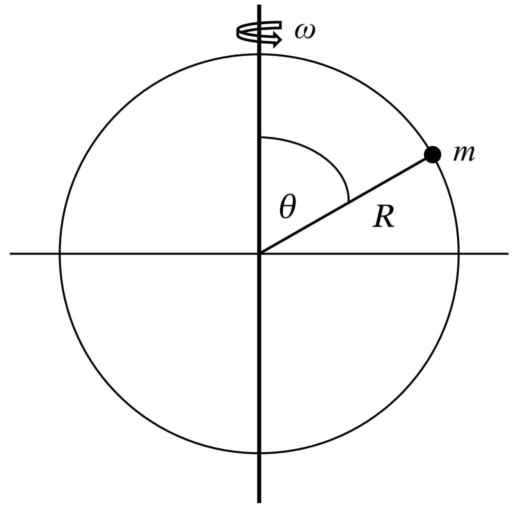
1. A bead of mass m slides freely on a circular wire of radius R. The wire rotates about a vertical diameter with constant angular velocity, ω. Let the position of the mass have spherical polar coordinates (r, θ, ϕ), where θ is the angle between the position vector and the vertical rotation axis, as shown in the figure. By using two equations of constraint, find expressions for the force exerted by the wire on the bead.



Solution: The kinetic energy of a free particle in spherical polar co-ordinates is

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right).$$

The gravitational potential energy of the bead is

$$U = mgr \cos \theta.$$

Hence, the Lagrangian is

$$L = T - U = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) - mgr \cos \theta.$$

The two equations of constrained are

$$g_1 (r, \theta, \phi, t) = r - R,$$

and

$$g_2 (r, \theta, \phi, t) = \phi - \omega t.$$

The Lagrange equations of motion in the presence of constraints are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i} + \sum_j \lambda_j (t) \frac{\partial g_j}{\partial q_i}.$$

Hence, the three equations of motion are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r} + \lambda_1 (t) \frac{\partial g_1}{\partial r} + \lambda_2 (t) \frac{\partial g_2}{\partial r},$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta} + \lambda_1 (t) \frac{\partial g_1}{\partial \theta} + \lambda_2 (t) \frac{\partial g_2}{\partial \theta},$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} + \lambda_1 (t) \frac{\partial g_1}{\partial \phi} + \lambda_2 (t) \frac{\partial g_2}{\partial \phi},$$

which give

$$
\begin{aligned}
&\frac{d}{dt}\left(m\dot{r}\right) = m\left(r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2\right) - mg\cos\theta + \lambda_1\left(t\right),\\
&\frac{d}{dt}\left(mr^2\dot{\theta}\right) = mr^2\sin\theta\cos\theta\dot{\phi}^2 + mgr\sin\theta,\\
&\frac{d}{dt}\left(mr^2\sin^2\theta\dot{\phi}\right) = \lambda_2\left(t\right).
\end{aligned}
\tag{1}
$$

On applying the equations of constraint, these equations become

$$
\begin{aligned}
&\lambda_1\left(t\right) = mg\cos\theta - m\left(R\dot{\theta}^2 + R\sin^2\theta\omega^2\right),\\
&\ddot{\theta} = \sin\theta\cos\theta\omega^2 + \frac{g}{R}\sin\theta,\\
&\lambda_2\left(t\right) = 2mR^2\omega\sin\theta\cos\theta\dot{\theta}.
\end{aligned}
$$

The generalized forces of constraint are

$$
Q_i = \sum_j \lambda_j\left(t\right)\frac{\partial g_j}{\partial q_i}.
$$

Hence

$$
\begin{aligned}
&Q_r = \lambda_1\left(t\right) = mg\cos\theta - m\left(R\dot{\theta}^2 + R\sin^2\theta\omega^2\right),\\
&Q_\theta = 0,\\
&Q_\phi = \lambda_2\left(t\right) = 2mR^2\omega\sin\theta\cos\theta\dot{\theta}.
\end{aligned}
$$

By inspection of the equations of motion (1), we see that Q_r is the radial component of the normal force that is exerted by the wire of the bead. Similarly, by recognizing that $mr^2\sin^2\theta\dot{\phi}$ is the component of the angular momentum of the bead parallel to the axis of rotation of the wire, we see that Q_ϕ is a torque about the axis of rotation of the wire due to the other component of the normal force, which must have magnitude

$$
N_\phi = \frac{Q_\phi}{R\sin\theta} = 2mR\omega\cos\theta\dot{\theta}.
$$

As a check, consider an equilibrium position for the case, $R\omega^2 \geq g$. We have $R\omega^2\cos\theta = -g$. Then

$$
Q_r = mg\cos\theta - m\sin^2\theta R\omega^2 = mg\cos\theta + mg\frac{\sin^2\theta}{\cos\theta} = \frac{mg}{\cos\theta},
$$

which agrees with what is obtained by applying Newton's laws of motion (consider the component of the normal force that is required to balance the force of gravity).

2. Consider a bead of mass *m* sliding without friction on a wire that is bent into the shape of a parabola and spun with constant angular velocity ω about its vertical axis. Use cylindrical polar coordinates and let the equation of the parabola be $z = k\rho^2$. Write down the Lagrangian in terms of ρ as the generalized coordinate. Find the equation of motion of the bead and determine whether there are positions of equilibrium. Discuss the stability of an equilibrium positions you find.

Solution: The kinetic and potential energies are

$$T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\omega^2 + \dot{z}^2\right) = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\omega^2 + 4k^2\rho^2\dot{\rho}^2\right),$$

$$U = mgz = mgk\rho^2.$$

Hence the Lagrangian is

$$L = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\omega^2 + 4k^2\rho^2\dot{\rho}^2\right) - mgk\rho^2.$$

The equation of motion is

$$\frac{d}{dt}\left[m\left(1 + 4k^2\rho^2\right)\dot{\rho}\right] = m\left(\rho\omega^2 + 4k^2\rho\dot{\rho}^2\right) - 2mgk\rho.$$

This simplifies to

$$\left(1 + 4k^2\rho^2\right)\ddot{\rho} = \rho\omega^2 - 4k^2\rho\dot{\rho}^2 - 2gk\rho.$$

The condition for equilibrium is that all the time derivatives are zero. Hence equilibria occur at $\rho = 0,$ and when $\omega^2 = 2gk.$

To test $\rho = 0$ for stability, we expand the equation of motion about this point. The linearized equation is

$$\ddot{\rho} = \left(\omega^2 - 2gk\right)\rho.$$

From this we see that stability requires

$$\omega^2 < 2gk.$$

When $\omega^2 = 2gk,$ the equation of motion is

$$\left(1 + 4k^2\rho^2\right)\ddot{\rho} = -4k^2\rho\dot{\rho}^2.$$

This can be written as

$$\frac{\ddot{\rho}}{\dot{\rho}} = -\frac{4k^2\rho\dot{\rho}}{1 + 4k^2\rho^2},$$

which can be integrated with respect to time to get

$$\ln\dot{\rho} = -\frac{1}{2}\ln\left(1 + 4k^2\rho^2\right) + C,$$

where *C* is a constant of integration. Hence

$$\dot{\rho} = \frac{A}{\sqrt{\left(1 + 4k^2\rho^2\right)}},$$

where A is also constant. We see that if $A > 0$, then $\dot{\rho}$ is always positive and hence $\rho \to \infty$. Similarly, if $A < 0$, then $\dot{\rho}$ is always negative and hence $\rho \to 0$. We conclude that the equilibrium is unstable with the possible exception of $\rho = 0$. The stability of the equilibrium point at $\rho = 0$ depends on the nature of the perturbation. If the perturbation has an initial $\dot{\rho} > 0$, then the equation above shows that $\dot{\rho} > 0$ for all time and the mass does not return to the equilibrium point. We have to conclude that $\rho = 0$ is an unstable equilibrium.

Alternatively, since the Lagrangian does not explicitly depend on t, the Hamiltonian is conserved. The Hamiltonian is

$$H = \dot{\rho}\frac{\partial L}{\partial \dot{\rho}} - L = \frac{1}{2}m\left(1+4k^2\rho^2\right)\dot{\rho}^2 + \left(gk - \frac{1}{2}\omega^2\right)m\rho^2.$$

By considering the last term on the right hand side as an effective potential, we reach the same conclusions as above.

3. Let $F = F\left(q_1,\cdots,q_n\right)$ be any function of the generalized coordinates of a system with Lagrangian $L\left(q_1,\cdots,q_n,\dot{q}_1,\cdots,\dot{q}_n,t\right)$. Prove that L and $L' = L + dF/dt$ give exactly the same equations of motion.

Solution: The equations of motion obtained from L' are

$$0 = \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) - \frac{\partial L'}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\frac{dF}{dt}\right) - \frac{\partial}{\partial q_i}\frac{dF}{dt}\right]. \tag{2}$$

Since F does not depend explicitly on time and the generalized velocities

$$\frac{dF}{dt} = \sum_k \frac{\partial F}{\partial q_k}\dot{q}_k.$$

Hence

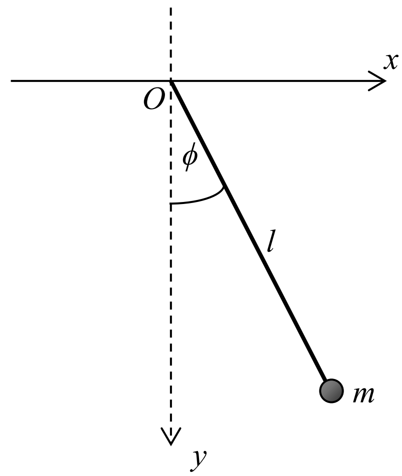
$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\frac{dF}{dt}\right) - \frac{\partial}{\partial q_i}\frac{dF}{dt} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\sum_k \frac{\partial F}{\partial q_k}\dot{q}_k\right) - \frac{\partial}{\partial q_i}\sum_k \frac{\partial F}{\partial q_k}\dot{q}_k$$
$$= \frac{d}{dt}\frac{\partial F}{\partial q_i} - \sum_k \frac{\partial^2 F}{\partial q_i \partial q_k}\dot{q}_k.$$

Since the partial derivatives of F are also simply functions of the generalized coordinates, we have

$$\frac{d}{dt}\frac{\partial F}{\partial q_i} - \sum_k \frac{\partial^2 F}{\partial q_i \partial q_k}\dot{q}_k = \sum_k \frac{\partial^2 F}{\partial q_k \partial q_i}\dot{q}_k - \sum_k \frac{\partial^2 F}{\partial q_i \partial q_k}\dot{q}_k.$$

Assuming that the function F is sufficiently well behaved that the order of partial differentiation is interchangeable, the right hand side is zero and hence the terms inside the square brackets in equation (2) sum to zero. We conclude that L and L' give exactly the same equations of motion.

4. Write down the Lagrangian for the simple pendulum in terms of the rectangular coordinates x and y. These coordinates are constrained to satisfy the equation $f(x,y)=\sqrt{x^2+y^2}-l=0.$



(a) Write down the two modified Lagrange equations. Comparing these with the two components of Newton's second law, show that the Lagrange multiplier is (minus) the tension in the rod. Verify that this is consistent with the expressions for the generalized force.

(b) The constraint equation can be written in many different ways. For example, we could have written $g(x,y)=x^2+y^2-l^2=0.$ Check that this function would have given the same physical results.

Solution: The Lagrangian is

$$L=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)+mgy.$$

(a) The Lagrange equations of motion with the constraint are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)=\frac{\partial L}{\partial q}+\lambda\frac{\partial f}{\partial q},$$

where λ is the Lagrange multiplier.

The $x-$ equation is

$$\frac{d}{dt}(m\dot{x})=\lambda\frac{x}{\sqrt{x^2+y^2}}=\lambda\sin\phi,$$

and the $y-$ equation is

$$\frac{d}{dt}(m\dot{y})=\lambda\frac{y}{\sqrt{x^2+y^2}}+mg=\lambda\cos\phi+mg.$$

The two Newtonian equations are

$$m\ddot{x}=-T\sin\phi,$$

and

$$m\ddot{y}=-T\cos\phi+mg,$$

where T is the tension in the rod. Hence, we see that the Lagrange multiplier is indeed minus the tension.

The generalized forces of constraint are

$$Q_x = \lambda \frac{\partial f}{\partial x} = -T \frac{x}{\sqrt{x^2 + y^2}} = -T \sin\phi,$$

and

$$Q_y = \lambda \frac{\partial f}{\partial y} = -T \frac{y}{\sqrt{x^2 + y^2}} = -T \cos\phi,$$

which is consistent with Newton's second law.

(b) Repeating the analysis with the constraint in the form $g(x, y) = x^2 + y^2 - l^2 = 0,$ we find that the equations of motion are

$$\frac{d}{dt}(m\dot{x}) = 2\lambda x,$$

and

$$\frac{d}{dt}(m\dot{y}) = 2\lambda y + mg.$$

We see that the Lagrange multiplier is now $\lambda = -T/2l$, and not simply the tension in the rod.

The generalized forces of constraint are now

$$Q_x = \lambda \frac{\partial g}{\partial x} = -\frac{T}{2l} 2x = -T \sin\phi,$$

and

$$Q_y = \lambda \frac{\partial g}{\partial y} = -\frac{T}{2l} 2y = -T \cos\phi.$$

These are same as above and hence the same physical results are obtained.