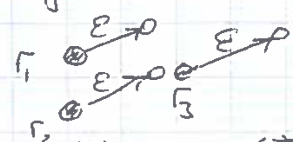
L35 Conservation Laws, Noether's theorem

* translation invariance $\Rightarrow$ conservation of $\vec{P}$

consider isolated system of N particles.
introduce arbitrary translation in space

$$\vec{r}_i \to \vec{r}_i + \vec{\varepsilon}$$



$$U(\vec{r}_i) \to U(\vec{r}_i + \vec{\varepsilon}) \qquad \delta U = U(\vec{r}_i+\vec{\varepsilon}) - U(\vec{r}_i) = 0$$

$$\vec{v}_i = \dot{\vec{r}}_i + \dot{\vec{\varepsilon}} = \dot{\vec{r}}_i \quad \text{so } \delta T = 0 \quad \& \quad \delta\mathcal{L} = 0$$

Consider just one coordinate $\{x_i\} = \{x_1, x_2, .. x_N\}$

$$\delta\mathcal{L} = \varepsilon \sum \frac{\partial \mathcal{L}}{\partial x_i} = 0 \to \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial x_i} = 0$$

But $\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{d}{dt} p_{x_i}$

$$\sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial x_i} = \sum_{i=1}^{N} \frac{d}{dt} p_{x_i} = \frac{d}{dt}\sum_{i=1}^{N} p_{x_i} = \frac{dP_x}{dt} = 0 \qquad P_x = \text{const.}$$

$$\text{or} \quad \sum \vec{P}_i = P_{tot} = \text{const.}$$

* translation in time $\Rightarrow$ conservation of energy.

$$\frac{d}{dt}\mathcal{L}(q_i, \dot{q}_i, t) = \sum_i \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t}$$

$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} p_i = \dot{p}_i$ time derivative of generalized momentum.

$$\frac{d}{dt}\mathcal{L} = \sum_i (\dot{p}_i \dot{q}_i + p_i \ddot{q}_i) + \frac{\partial \mathcal{L}}{\partial t} = \frac{d}{dt}\sum_i (p_i \dot{q}_i) + \frac{\partial}{\partial t}\mathcal{L}$$

$$\text{if } \frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \underbrace{\sum_i p_i \dot{q}_i - \mathcal{L}}_{\text{Hamiltonian } H \text{ is conserved.}} = \text{const}$$

no explicit dependence on t

natural coordinates: $\vec{r}_\alpha = \vec{r}_\alpha(q_1, q_2 .. q_N)$ time independent

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha^{\,2} \qquad \dot{\vec{r}}_\alpha = \sum_{i=1}^{N} \frac{\partial \vec{r}_\alpha}{\partial q_i}\dot{q}_i$$

$$T = \frac{1}{2}\sum_\alpha m_\alpha \left[\sum_i \frac{\partial r_\alpha}{\partial q_i}\dot{q}_i \sum_j \frac{\partial r_\alpha}{\partial q_j}\dot{q}_j\right] = \frac{1}{2}\sum_{ij} A_{ij}\dot{q}_i \dot{q}_j$$

L 35 $$A_{ij} = \sum_{\alpha} m_\alpha \left(\frac{\partial \vec{r}_\alpha}{\partial q_i}\right) \cdot \left(\frac{\partial \vec{r}_\alpha}{\partial q_j}\right)$$

$$T = \frac{1}{2} \sum_{ij} A_{ij} \dot{q}_i \dot{q}_j \qquad \frac{\partial T}{\partial q_i} = \frac{2}{2} \sum_j A_{ij} q_j = p_i$$

$$\sum_{i=1}^{N} p_i \dot{q}_i = \sum_i \dot{q}_i \sum_j A_{ij} \dot{q}_j = \sum_{ij} A_{ij} \dot{q}_i \dot{q}_j = 2T$$

$$H = \sum_{i=1}^{N} p_i \dot{q}_i - \mathcal{L} = 2T - (T - U) = T + U$$

when generalized coordinates are natural

H = mechanical Energy.