L30 EL equation for unconstrained motion

particle moving in 3D $T = \frac{1}{2} m \dot{\vec{r}}^2$

potential energy $U = U(\vec{r}) - U(x,y,z)$

The Lagrangian is: $\mathcal{L} = T - U$

$\mathcal{L} = \mathcal{L}(x, \dot{x}, y, \dot{y}, z, \dot{z})$ $\quad x = x(t) \quad y = y(t) \quad z = z(t)$

$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x \qquad \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} = p_x$

N. second Law $F_x - \dot{p}_x = \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$

N second Law is equivalent to EL equation.

* Hamilton principle

action Integral $A = \int_1^2 \mathcal{L} dt$

the path which a particle follows between points 1 & 2 in a given time interval $t_1 \div t_2$ is such that the action integral is stationary along that path.

$\vec{F} = m\vec{a} \equiv \frac{\partial \mathcal{L}}{\partial x_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} \equiv$ $x,y,z(t)$ determined by stat. action integral.

* generalized coordinates $q_i = q_i(\vec{r}) \quad i=1,2,3$

$\vec{r} = \vec{r}(q_1, q_2, q_3)$

The Lagrangian is $\mathcal{L}(q_1, \dot{q}_1, q_2 \dot{q}_2, q_3 \dot{q}_3)$

$A = \int_1^2 \mathcal{L}(q_i \dot{q}_i) \, dt \rightarrow \frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$

* Particle in 2D

case (a) Cartesian coordinates.

$\mathcal{L} = T - U = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - U(x,y)$

$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \rightarrow F_x = m\ddot{x}$

$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{y}} \rightarrow F_y = m\ddot{y}$

L30 Case (b) Polar coordinates.

$$T - U = \frac{1}{2}m(\dot r^2 + r^2\dot\varphi^2) - U(r)$$

$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot\varphi^2 - \frac{\partial U}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot r}\right) = \frac{d}{dt}(m\dot r) = m\ddot r$$

$$-\frac{\partial U}{\partial r} = F_r = m(\ddot r - r\dot\varphi)$$

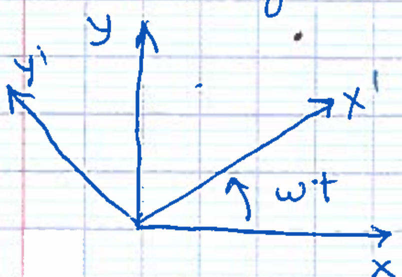
$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\varphi} \rightarrow -\frac{\partial U}{\partial \varphi} = \frac{d}{dt}(mr^2\dot\varphi)$$

$$\nabla_{r\varphi} U = \frac{\partial U}{\partial r}\hat r + \frac{1}{r}\frac{\partial U}{\partial \varphi}\hat\varphi \qquad F_\varphi = -\frac{1}{r}\frac{\partial U}{\partial \varphi}$$

$r F_\varphi = \Gamma$ - torque $\rightarrow F_\varphi r = \Gamma = \frac{d}{dt}(mr^2\dot\varphi)$

or $\Gamma = \frac{dL}{dt}$ $\quad L = mr^2\dot\varphi$ – angular momentum

* rotating coordinates.



$$T = \frac{m}{2}(\dot x^2 + \dot y^2) \qquad U(x,y) = 0$$

$$x = x'\cos\omega t + y'\sin\omega t$$
$$y = y'\cos\omega t - x'\sin\omega t$$

$$\dot x = \dot x'\cos\omega t + \dot y'\sin\omega t - \omega x'\sin\omega t + y'\omega\cos\omega t =$$
$$= (\dot x' + y'\omega)\cos\omega t + (\dot y' - x'\omega)\sin\omega t$$
$$\dot y = (\dot y' - x'\omega)\cos\omega t - (\dot x' + y'\omega)\sin\omega t$$
$$\dot x^2 + \dot y^2 = (\dot x' + y'\omega)^2 + (\dot y' - x'\omega)^2$$

$$\mathcal{L} = \underbrace{\frac{1}{2}m(\dot x'^2 + \dot y'^2)}_{T} + \underbrace{\frac{1}{2}m\omega^2(x'^2 + y'^2)}_{-\,U_c} + \underbrace{m\omega(\dot x'y' - \dot y'x')}_{+\text{ Coriolis term}}$$

$U_c = -\frac{1}{2}m\omega^2(x'^2 + y'^2)$ – "potential energy" due to centrifugal force

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot x} = \frac{d}{dt}(m\dot x' + m\omega y') = m\omega^2 x' - m\omega\dot y'$$

$$m\ddot x' = \underbrace{m\omega^2 x'}_{\text{centrifugal}} - \underbrace{2m\omega\dot y'}_{\text{Coriolis}}$$