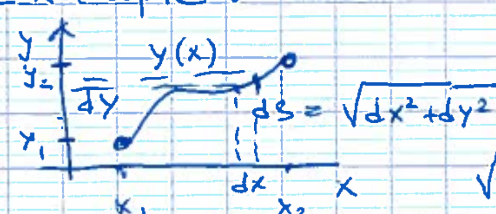
L28 * CM - rules of motion for a given system.
Once we set boundary conditions, CM will predict the system motion in the future (and in the past)
These stationary trajectories can be obtained by variation of a functional, which describes all the phycics of a given system

* Calculus of variation
define integral $S = \int_{x_1}^{x_2} f(y(x), y'(x), x)\, dx$
in physics S is called action and f is Lagrangian.
For now just consider a mathematical/probl
- What is the function $y(x)$ which make $S(y)$ stationary - eq. smal variations in $y(x)$ leave S unchanged.
or $\delta(S) = 0$ (variation of $S = 0$)

* Example: shortest path between 2 points



$$S = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{dx^2+dy^2} = \int_{x_1}^{x_2} \sqrt{1+y'^2(x)}\, dx$$

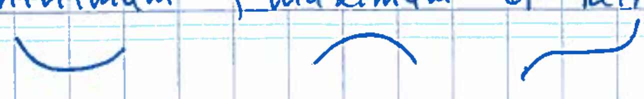
$$\sqrt{dx^2+dy^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$$

* Example: Fermat principle
path of light between 2 pts:
$v = c/n(x,y)$ $\quad T = \int_1^2 dt = \int_1^2 \frac{ds}{v} = \int_1^2 \frac{1}{c} n\, ds$

$$T = \frac{1}{c}\int_1^2 n(x,y)\sqrt{1+y'^2}\, dx$$

Variation problem: find $y(x)$ that makes T minimum.

* Stationary trajectory $y(x)$: $\delta S(y(x)) = 0$
eq. S is minimum, maximum or neither.



L28 The Euler – Lagrange Equation.

$$\delta\left[S = \int_{x_1}^{x_2} f[y(x), y'(x), x]\,dx\right] = 0$$

let say $y(x)$ is a solution $\delta(S(y_0)) = 0$

define $Y(x) = y_0(x) + \alpha\eta(x)$ where

α = some constant $\quad \eta(x_1) = \eta(x_2) = 0$

$\eta(x)$ – arbitrary function

** Derivation ① $\quad S(\alpha) = \int_{x_1}^{x_2} f(Y, Y', x)\,dx$

$$Y'(x) = y_0'(x) + \alpha\eta'(x)$$

$$\frac{dS}{d\alpha} = \int_1^2 \frac{df}{d\alpha}\,dx = \int_{x_1}^{x_2}\left(\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}\right)dx = 0$$

$$\int_{x_1}^{x_2}\eta'\frac{\partial f}{\partial y'}\,dx = \underbrace{\eta(x)\frac{\partial f}{\partial y'}\Big|_{x_1}^{x_2}}_{=0} - \int_{x_1}^{x_2}\eta(x)\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]dx$$

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2}\left[\eta\frac{\partial f}{\partial y} - \eta\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\right]dx = 0$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$$ because $\frac{\partial S}{\partial\alpha} = 0$ for ANY $\eta(x)$ and continuous $Y(x)$

Euler-Lagrange equation

** Derivation ②

$$S = \int_{x_1}^{x_2} f(y, y', x)\,dx = \lim_{\Delta\to 0}\Delta\sum_n f\left(y_n, \frac{y_{n+1}-y_n}{\Delta}, n\Delta\right)$$

for discrete $x_n = n\Delta$, $\quad y_n' = \dfrac{y_{n+1}-y_n}{\Delta}$

pick arbitrary point n and vary y_n

only 2 terms of the sum depend on y_n

$$\Delta S = \Delta\left[f\left(y_n, \frac{y_{n+1}-y_n}{\Delta}, n\Delta\right) + f\left(y_{n-1}, \frac{y_n - y_{n-1}}{\Delta}, (n-1)\Delta\right)\right]$$

$$\frac{\partial\Delta S}{\partial y_n} = \Delta\frac{\partial f}{\partial y_n} - \Delta\frac{\partial f}{\partial y_n'}\frac{1}{\Delta} + \Delta\frac{\partial f}{\partial y_{n-1}'}\frac{1}{\Delta} = 0$$

$$\Delta\left[\frac{\partial f}{\partial y_n} - \frac{1}{\Delta}\left(\frac{\partial f}{\partial y_n'} - \frac{\partial f}{\partial y_{n-1}'}\right)\right] = 0$$

$$\Rightarrow \frac{\partial f}{\partial y_n} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_n'}\right) = 0$$ when applied to any point n

$$\Rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$ Euler-Lagrange equation.