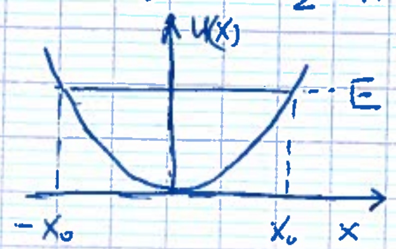
220 Hooke's law : $F_x(x) = -kx$

conservative force → potential energy $U(x) = \frac{kx^2}{2}$

Consider arbitrary $U(x)$ and expand it in Tylor series around $x=0$ (equilibrium)

$$U(x) = \underbrace{U(0)}_{const=0} + \underbrace{\dot{U}(0)}_{=0 \text{ at equilibrium}} x + \frac{1}{2}\underbrace{\ddot{U}(0)}_{k} x^2$$

$U(x) = \frac{1}{2}kx^2 + O(x^3)$ for any local minima for $U(x)$



$$E = \frac{m\dot{x}^2}{2} + \frac{1}{2}kx^2$$

$$\frac{dE}{dt} = 0 \rightarrow m\dot{x}\ddot{x} + kx\dot{x} = 0$$

* Simple harmonic motion

2nd Newton's law : $m\ddot{x} = -kx$ →

$$\ddot{x} = -\frac{k}{m}x = -\omega^2 x \qquad \omega = \sqrt{k/m}$$

\- Exponential solutions:

$x_1(t) = e^{i\omega t}$ $\quad x_2 = e^{-i\omega t}$ → $x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$

superposition of 2 solutions is also a solution. general solution

\- Sin & Cos use Euler's formula.

$$e^{\pm i\omega t} = \cos\omega t \pm i\sin\omega t$$

$$x(t) = \underbrace{(C_1 + C_2)}_{B_1}\cos\omega t + \underbrace{i(C_1 - C_2)}_{B_2}\sin\omega t \rightarrow$$ simple harmonic motion

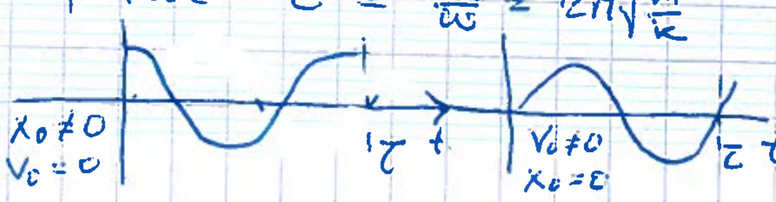
B_1 & B_2 are real

\- initial conditions : $B_1 = x_0$ $\quad (x(0) = B_1)$

$0 \neq x_0, v_0 = 0 \rightarrow x = x_0\cos\omega t$

$x_0 = 0, v_0 \neq 0 \rightarrow x(t) = \frac{v_0}{\omega}\sin\omega t \qquad v_0 = \omega B_2$

period $\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$



L20 Phase shifted cos solution.

define $A = \sqrt{B_1^2 + B_2^2}$

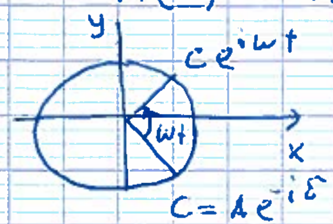
$$x(t) = A\left[\frac{B_1}{A}\cos\omega t + \frac{B_2}{A}\sin\omega t\right] = A[\cos\delta\cos\omega t + \sin\delta\sin\omega t]$$
$$= A\cos(\omega t - \delta) \qquad tg\,\delta = B_2/B_1$$

— Real part of complex Exponential

$$C_1 = \frac{1}{2}(B_1 - iB_2) \qquad C_2 = \frac{1}{2}(B_1 + iB_2)$$
$$C_2 = C_1^* \quad \rightarrow \quad x(t) = \underbrace{C_1 e^{i\omega t}}_{z} + \underbrace{C_1^* e^{-i\omega t}}_{z^*}$$

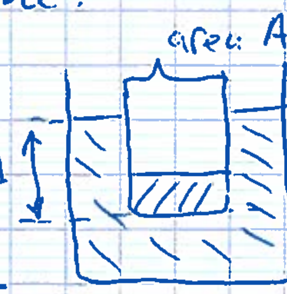
$$z + z^* = (x + iy) + (x - iy) = 2x = 2\,\mathrm{Re}\,z$$
$$x(t) = 2\,\mathrm{Re}\{C_1 e^{i\omega t}\}$$

$$2C_1 = B_1 - iB_2 = A[\cos\delta - i\sin\delta] = Ae^{-i\delta}$$
$$x(t) = 2\,\mathrm{Re}\{C_1 e^{i\omega t}\} = A\,\mathrm{Re}\{e^{i(\omega t - \delta)}\}$$



$x(t) = A\cos(\omega t - \delta)$ — oscillation along x can be presented as a rotation of a vector in xy plane.

— Example 5.2



$$mg = \rho A d_0 g \rightarrow \text{Archimede's force}$$

$$m\ddot{x} = mg - \rho g A(d_0 + x) =$$
$$\ddot{x} = -g/d_0\, x \qquad \omega = \sqrt{g/d_0}$$

— Oscillator energy: $E = \frac{m\dot{x}^2}{2} + \frac{1}{2}kx^2$

$$T = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t - \delta)$$
$$U = \frac{1}{2}kA^2\cos^2(\omega t - \delta)$$
$$E = T + U = \frac{1}{2}kA^2$$