L12 Angular Momentum of Several Particles.

* $\vec{L} = \sum_{\alpha=1}^{N} \ell_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha \qquad \dot{\vec{L}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$

separate "internal" and "external" forces:

$$\vec{F}_\alpha = \sum_{\alpha \neq \beta} \vec{F}_{\alpha\beta} + \vec{F}^{\alpha}_{ext}$$

$$\dot{\vec{L}} = \sum_\alpha \sum_{\beta \neq \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \sum_\alpha \vec{r}_\alpha \times F^{\alpha}_{ext}$$

for each $\alpha\beta$ pair write: $\vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \vec{r}_\beta \times \vec{F}_{\beta\alpha}$

using the third law: $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$

so $(\vec{r}_\alpha - \vec{r}_\beta) \times \vec{F}_{\alpha\beta} = 0$ for

the central force $\vec{F}_{\alpha\beta} \parallel$ to $(\vec{r}_\alpha - \vec{r}_\beta)$

$$\dot{\vec{L}} = \sum_\alpha \vec{r}_\alpha \times F^{\alpha}_{ext} = \sum \vec{\Gamma}^{\alpha}_{ext} = \vec{\Gamma}_{ext}$$

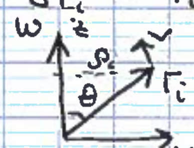
— Conservation of angular momentum: $\dot{\vec{L}} = 0$ if $\vec{\Gamma}_{ext} = 0$

* Moment of Inertia

consider a motion (rotation) of a rigid body – individual particles do not move wrt each other.

a) symmetric wrt the rotation axis

$d\vec{\ell}_i = \vec{r}_i \times [\vec{\omega} \times \vec{r}_i] dm_i =$



$d\ell^i_z = \omega r_i^2 \sin^2\theta \, dm =$

$= \omega \rho_i^2 dm$

$d\ell^i_x = -\omega r_i^2 \sin\theta \cos\theta \, dm$

When integrate over all masses

$$L_z = \sum dm_i \rho_i^2 \omega = I\omega \qquad L_x = 0$$

Moment of Inertia $I = \sum_i m_i \rho_i^2$

ρ_i – distanse to the rotation axis.

sample 3.3

mass per unit area $= M/\pi R^2 = dm/dA$.

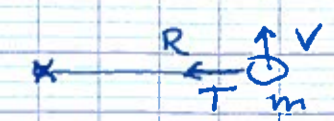
$\Delta m = 2\pi r \, dr \cdot \frac{M}{\pi R^2} \qquad \Delta I = \Delta m \cdot r^2$

$$I = \int_0^R \frac{M}{R^2} 2r^3 dr = \frac{M}{R^2} R^4/2 = M R^2/2$$

$$L_z = mvb = (m + \tfrac{M}{2}) R^2 \omega \qquad w = \frac{m}{m + M/2} \frac{v b}{R^2}$$

* Moment of inertia & conservation of L

a. mass on string



T – string tension

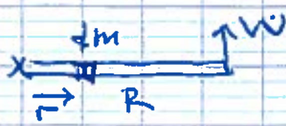
$\vec{L} = \vec{R} \times \vec{P}$ $\quad \vec{P} = m\vec{v}$

$\dot{\vec{L}} = \vec{R} \times \vec{T} = 0$ $\quad \omega$

$L = Rmv = mR^2 \frac{v}{R}$;

$L = I\omega$ $\quad I = mR^2$ – moment of inertia of the mass m rotating around the axis at r=0

b uniform rod



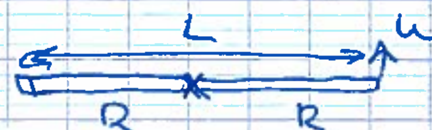
$M = \rho R$ – mass

$dI = dm \cdot r^2 = dr \rho r^2 \rightarrow I = \int dI = \int \rho r^2 dr$

$I = \rho \frac{r^3}{3} \Big|_0^R = \rho R \frac{R^2}{3} = M \frac{R^2}{3}$

$L = M \frac{R^2}{3} \omega$

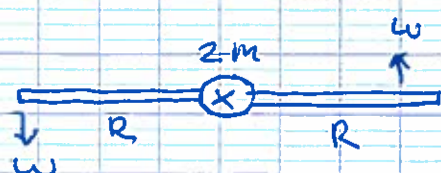
c. symmetric rod



$M = \rho L$

$I = 2\rho R \frac{R^2}{3} = M \frac{R^2}{3} = M \frac{L^2}{12}$ $\quad L = 2R$

d. rod + mass



$M_{rod} = \rho L$

2m = M_{rod}.

$I_{2m} = 2m \bullet 0 = 0$

$I_{rod} = M \frac{R^2}{3}$

$L = M \frac{R^2}{3} \omega$

c rod + mass – Let masses 2m move freely along the rod, so they move to the ends of the rod under influence of the centrifugal force.

$I_{2m} = 2mR^2$ $\quad I_{rod} = M \frac{R^2}{3} \rightarrow L' = M \frac{4}{3} R^2 \omega'$

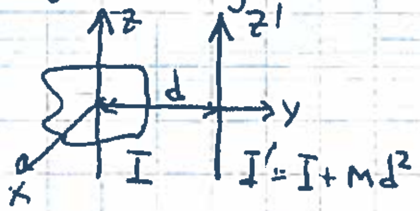
ω' – new angular velocity $\omega' = \omega/4$

since $L' = L$

L12

Parallel axis (Huygens-Steiner) theorem

Defines moment of inertia of a rigid body about any parallel axis.

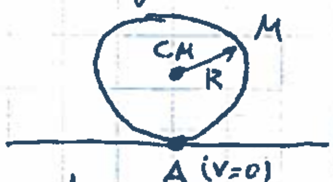


$$I = \sum dm_i \rho_i^2 = \sum dm_i (x_i^2 + y_i^2)$$

$$I' = \sum dm_i [x_i^2 + (y_i + d)^2] =$$

$$= \underbrace{\sum dm_i (x_i^2 + y_i^2)}_{I} + \underbrace{\sum dm_i d^2}_{Md^2} + \underbrace{\sum dm_i y d}_{y_{cm} = 0}$$

Rolling wheel (not slipping)



$I_{cm} = MR^2$ (about CM)

$I_A = MR^2 + MR^2 = 2MR^2$ (about A)

2nd N's law for rotation is $\dot{\vec{L}}_{cm} = \vec{\Gamma}_{cm}$ even if CM is not an inertial ref. frame.

$$\dot{L}_{cm} = I_{cm} \dot{\omega} = \Gamma_{cm}$$

apply force F to CM

Newton's laws:

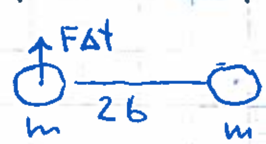
① $m\dot{v} = F - f$ f - friction force

② $MR^2\dot{\omega} = fR$

F torque about CM = 0

$f = m\dot{v} = F - f \quad f = F/2$

Example 3.4 from textbook



$\vec{F} \cdot \Delta t = \vec{p}$

$\vec{P} = 2m\dot{\vec{R}}_{cm} \rightarrow V_{cm} = \dot{\vec{R}} = \vec{F}\Delta t / 2m$

$|\vec{\Gamma}| = Fb = \dot{L} \rightarrow L = Fb\Delta t = I\omega$

$I = 2mb^2 \rightarrow \omega = \dfrac{Fb\Delta t}{2mb^2}$

in CM both masses move with $V = \omega b$

in lab frame $V_L = V_{cm} + \omega b = \dfrac{F\Delta t}{m}$

$V_R = V_{cm} - \omega b = 0$