L7

Trajectory & Range for $\vec{F} = m\vec{g} - b\vec{V}$

$x(t) = V_{x0}\tau\left(1 - e^{-t/\tau}\right)$ $\qquad V_T = g\tau$

$y(t) = (V_{y0} + V_T)\tau\left(1 - e^{-t/\tau}\right) - V_T t$ $\qquad \tau = m/b$

– trajectory $y(x) = \dfrac{V_{y0} + V_T}{V_{x0}} x + V_T\tau \ln\left(1 - \dfrac{x}{V_{x0}\tau}\right)$

– Horizontal range R: $\dfrac{V_{y0} + V_T}{V_{x0}} R + V_T\tau \ln\left(1 - \dfrac{R}{V_{x0}\tau}\right) =$

– Weak linear drag approximation: b – small, τ – large.

$V_T \gg V_{x0}, V_{y0}$; $R \ll V_{x0}\tau$;

use $\boxed{\varepsilon = \dfrac{R}{V_{x0}\tau}}$

use Taylor's expansion of $\ln(1-\varepsilon)$

$$f(\varepsilon) = f(0) + \left.\frac{df}{d\varepsilon}\right|_{\varepsilon=0}\varepsilon + \frac{1}{2!}\left.\frac{d^2f}{d\varepsilon^2}\right|_{\varepsilon=0}\varepsilon^2 + \frac{1}{3!}\left.\frac{d^3f}{d\varepsilon^3}\right|_{\varepsilon=0}\varepsilon^3 \ldots$$

$$\ln(1-\varepsilon) = 0 - \varepsilon - \frac{1}{2}\varepsilon^2 - \frac{1}{3}\varepsilon^3 \ldots$$

rewrite the equation for ε:

$$(V_{y0} + V_T)\tau\varepsilon - V_T\tau\left[\varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^2\right] = 0$$

trivial solution: $\varepsilon = 0 \rightarrow R = 0$

$$\varepsilon\tau \cdot \left[V_{y0} - \frac{V_T\varepsilon}{2} - \frac{V_T\varepsilon^2}{3}\right] = 0$$

$$\varepsilon = \frac{2V_{y0}}{V_T} - \frac{2}{3}\varepsilon^2 \rightarrow R = \underbrace{\frac{2V_{y0}V_{x0}}{g}}_{R_{vac}} - \frac{2}{3V_{x0}\tau}R$$

$$R = R_{vac} - \frac{2R^2}{3V_x\tau}$$

$R \rightarrow R_{vac}$ when $\tau \rightarrow \infty$

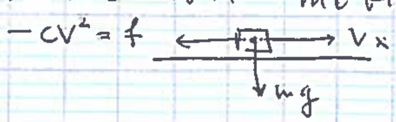
$\boxed{R \sim R_{vac}\left(1 - \frac{4}{3}\frac{V_{y0}}{V_T}\right.}$

* Quadratic Drag: $m\dot{\vec{V}} = m\vec{g} - cV^2\hat{V}$

non-liner diff. equation

no close form analyt. solution.

– Horizontal motion: $m\dot{V}_x = -cV_x^2$ $\qquad V_y = 0$



separation of variables:

$m\dot{x} = f(x) \rightarrow \dfrac{dx}{f(x)} = \dfrac{1}{m}dt$

L 7

$$\frac{dV_x}{V_x^2} = -\frac{c}{m}dt \rightarrow \int_{V_{0x}}^{V_x} \frac{dV}{V^2} = -\frac{c}{m}\int_0^t dt$$

$$\frac{1}{V_{0x}} - \frac{1}{V_x} = -\frac{c}{m}\tau \qquad V_x(t) = \frac{V_{0x}}{1+\frac{cV_{0x}}{m}t} = \frac{V_{0x}}{1+t/\tau}$$

$$X(t) = x_0 + \int_0^t v(t)\,dt = V_{0x}\tau \ln(1+t/\tau)$$

Important differenses between Liner & Quad

$$V_{lin} \sim e^{-t/\tau} \qquad V_{quad} \sim \frac{1}{t/\tau}$$

$$X_{lin} - \text{finite} \qquad X_{quad} \rightarrow \infty$$

– Vertical Motion: $m\dot{V}_y = mg - cV_y^2 \qquad V_x = 0$

terminal velocity: $V_T = \sqrt{\frac{mg}{c}}$

Solution: $\dot{V}_y = g\left(1 - \frac{V^2}{V_T^2}\right)$

$$\int_{V_{0y}}^{V} \frac{dV}{1 - V^2/V_T^2} = \int_0^t g t \rightarrow \frac{V_T}{g}\operatorname{arctanh}\left(\frac{V}{V_T}\right) = t$$

inverse hyperbolic tangent

$$V(t) = V_T \cdot \tanh\left(\frac{gt}{V_T}\right)$$

$$y(t) = \frac{V_T^2}{g}\ln\left[\cosh\left(\frac{gt}{V_T}\right)\right]$$

* baseball $V_T = \sqrt{\frac{mg}{\gamma D^2}} = \sqrt{\frac{0.15\,kg \cdot 9.8\,m/s^2}{0.25\,N\,s^2/m^4 \cdot 0.07^2\,m^2}} \approx 35\,m/s$

* "Spherical" human $V_T = \sqrt{\frac{60 \times 9.8}{0.25 \cdot (0.5)^2}} \sim 100\,m/s$

* baseball motion is dominated by drag force!
usuall $V > V_T$. for a good pitcher.

– Combined motion $m\ddot{\vec{r}} = m\vec{g} - c|V|\vec{V}$

$$m\dot{V}_x = -c\sqrt{V_x^2+V_y^2}\,V_x \rightarrow -c\frac{V_T}{m}V_x \quad t \rightarrow \infty$$

$$m\dot{V}_y = -mg - c\sqrt{V_x^2+V_y^2}\,V_y$$

no analytic solutio, but can be solved numerically.