

# Physical Quantities and Dimensions

## Physical Constants You Should Memorize.

$$\begin{aligned}\hbar &\approx 1 \times 10^{-34} \text{ Js } [ML^2T^{-1}] \\ c &\approx 3 \times 10^8 \text{ ms}^{-1} [LT^{-1}] \\ e &\approx 1.6 \times 10^{-19} \text{ C } [IT] \\ m_e &\approx 9 \times 10^{31} \approx 10^{-30} \text{ kg } [M] \\ \mu_o &= 4\pi \times 10^{-7} \text{ Hm}^{-1} [??] \\ \epsilon_o &= 1/(\mu_o c^2) \text{ Fm}^{-1} [??] \\ k_B &\approx 1.4 \times 10^{-23} \text{ JK}^{-1} [ML^2T^{-2}\theta^{-1}] \\ g &= 9.8 \text{ ms}^{-2} [LT^{-2}] \text{ (directly related to } G)\end{aligned}$$

## Base Quantity and Derived Quantity

Science including physics begins with observations of phenomena. Through rigorous and controlled experimentation and logical thought process, the physical phenomena are described quantitatively using mathematical tools. Any quantitative description of a property requires comparison with a reference, for example length needs a meter-stick. In this process we recognize a very obvious fact that properties of different kinds cannot be compared.

The allowed types of physical properties are called *Physical Quantities*. There are two classes of physical quantities: 1. Basic Quantities and 2. Derived Quantities. The physical base quantity of a certain kind  $\mathbf{A}$  can be represented by the multiplication of a number  $A$  and the unit quantity of the kind  $\hat{a}$ ,

$$\mathbf{A} = A\hat{a}$$

Given the type of base quantity, the choice of the unit is arbitrary; mass  $m$  can be in gram, kilogram, or pound. Obviously the numerical values of the quantity varies depending on the choice of the unit. However, the quantity itself does not change.

$$\mathbf{A} = A\hat{a} = A'\hat{a}'.$$

Here, we know if  $\hat{a}' = n\hat{a}$ , then  $A' = n^{-1}A$ . Therefore, it is crucial to know which unit of the physical quantity is used. For example, the length of an object can be written as 0.1 m or 3.937". If  $\mathbf{A}$  represents the length of the object,  $\hat{a}$  is 1 m, and  $\hat{a}'$  is 1", then  $A = 0.1$  and  $A' = 3.973$  with the knowledge of 1" = 0.0254 m.

There are only limited number of base quantities such as time, mass, length, temperature, angle, and so on. From these base quantities, one can generate in principle any physical quantities and they are called the derived quantities. Let us say that we measured the distance of travel  $\mathbf{L}$  of a vehicle and the time took  $\mathbf{t}$ . From the measurements, we can calculate a new quantity, the average speed of the travel,  $\mathbf{v} = \mathbf{L}/\mathbf{t}$ . So a derived quantity *speed* is defined. We can also calculate another derived (physical) quantity, kinetic energy  $E_k$  using another base quantity, mass  $M$  and the derived quantity speed:

$$\mathbf{E}_k = \frac{1}{2}\mathbf{m}\mathbf{v}^2.$$

Here I use bold face variables to emphasize that the quantities include proper units or unit independent. Below I make a very important statement without providing the proof.

*All monomial derived physical quantities have the following power-law form.*

$$Q = \alpha A^a B^b C^c \dots ,$$

where  $Q$  is the derived quantity and  $A, B, C, \dots$  are base quantities.  $\alpha$  is a numerical constant and  $a, b, c \dots$  are real numbers.

## Physical Dimension

Each type of base physical quantity has its own dimension. If  $A$  is the numerical value of a length, we say it has the dimension of *Length* regardless of the unit used:

$$[A] = L.$$

Each base quantity carries its own physical dimension as length ( $L$ ), mass ( $M$ ), time ( $T$ ),... See below for all base quantities. The dimension of a derived quantity  $Q = \alpha L^\ell M^m T^\tau \dots$  is given by

$$[Q] = (L)^\ell (M)^m (T)^\tau \dots$$

e. g. The dimension of kinetic energy is  $[E_k] = ML^2T^{-2}$  since the speed  $[v] = LT^{-1}$ . And all kinds of energy should have exactly the same dimension.

1. Any derived quantities involving special functions such as  $\ln$ ,  $\sin$ ,  $\exp$ , ... should have their argument **dimensionless**.

2. Physical equations must be **dimensionally homogeneous**: both sides of an equation should have the same dimension.

**EX:** The physical dimension of force is  $MLT^{-2}$ . Gravitational force between two objects is given by:

$$F_G = G \frac{m \cdot M}{r^2}.$$

What is the physical dimension of gravitational constant  $G$ ?

**EX:** Energy comes in many different forms: kinetic energy, gravitational energy, or work. No matter what it is, the physical dimension of those should be  $[E] = ML^2t^{-2}$ . Check this for the energies above.

## System of Units

What are the fundamental base quantities that one can describe physical phenomena properly? The system of units is composed of a complete set of base quantities, the base units, and all relevant derived quantities—of course they are open ended! There could be many different sets of system of units but Systeme International (SI) system is a unit system internationally adopted.

TABLE I: Dimensions of some commonly used physical quantities

Property	Symbol	B or D	Dimension	Unit
Length	$L$	B	$L$	meter (m)
Mass	$m$	B	$M$	kilogram (kg)
Time	$t$	B	$T$	second (s)
Temperature	$T$	B	$\theta$	kelvin (K)
Current	$I$	B	$I$	ampere (A)
Luminosity	$J$	B	$J$	candela (cd)
Number	$N$	B	1	mole (mol.)
Force	$F$	D	$MLT^{-2}$	newton (N)
Pressure	$P$	D	$ML^{-1}T^{-2}$	pascal (ps)
Energy	$E$	D	$ML^2T^{-2}$	joule (J)
Electric Charge	$Q$	D	$IT$	coulomb (C)
Electric Field	$E$	D	$MLT^{-3}I^{-1}$	
Magnetic Field	$H$	D	$IL^{-1}$	(A/m)

The SI system is composed of

6 base physical quantities: length, time, mass, temperature, current, luminous intensity + number (moles)  
and derived quantities.

## Important Usage of Physical Dimension

*Check if an equation is correct.*

Each term in an equation should have the same physical dimensions.

*Check if your answer is correct and physically meaningful.* You calculated an energy quantity. If your answer does not have the correct dimension, then your answer must be wrong!

*One can extract an expression without rigorous calculations.*

Here is an example. What is the period of a pendulum composed of a mass  $m$  and a string length of  $\ell$ ?

There is a systematic way of extracting a answer of a physics problem using physical dimension. It is called **Dimensional Analysis**. The following section provides a step-by-step process. This part is optional now. But I strongly encourage you to go over in the near future!

## Buckingham's $\Pi$ -Theorem (Optional)

The above examples give you an idea how to use Dimensional Analysis (DA) in expressing a physical quantities in terms of other physical quantities. These are the simplest kind of examples. There is a general principle that you can extract the dependencies in a systematic way: Buckingham's  $\Pi$  Theorem. At the heart of DA is the *dimensional homogeneity* of any physically significant equations relating various physical properties.

Let's consider a dependent physical quantity  $Q_o$ —this quantity depends on other physical quantities. Now the task is how to extract the specific dependence of each physical quantity in  $Q_o$ .

**Step 1:** Identify a complete set of independent quantities  $\{Q_1, Q_2, \dots, Q_n\}$  that determines  $Q_o$ :

$$Q_o = f(Q_1, Q_2, \dots, Q_N).$$

{*Independent Variables*} : the value of each member can be adjusted arbitrarily without affecting the values of others in the set.

For example, a set of variables describing the motion of an object  $E_k, v, m$  is not a set of independent variables because if you change the speed  $v$ , the kinetic energy of the object  $E_k$  also changes. So you can choose a set of independent variables as  $\{E_k, v\}$ ,  $\{E_k, m\}$ , or  $\{m, v\}$ . Anyone of those is a legitimate choice. Only using two variables, one can express all three.

## Step 2: Dimensional Consideration (Homogeneity)

One should know the dimension of all variables  $Q_o, Q_1, Q_2, \dots, Q_n$ :

$$[Q_i] = L^{\ell_i} M^{m_i} t^{\tau_i}.$$

Then, identify a subset of *dimensionally independent* variables  $\{Q_1, Q_2, \dots, Q_k\}$  from the set of the independent variables:

$$\{Q_1, Q_2, \dots, Q_n\} \longrightarrow \{Q_1, Q_2, \dots, Q_k\} \quad \{Q_{k+1}, \dots, Q_n\}.$$

The dimension of  $Q_{k+1}, \dots, Q_n$  can be expressed by a combination of  $\{Q_1, Q_2, \dots, Q_k\}$ . There two possible cases,  $k = n$  or  $k < n$ . If  $k = n$ , then all the independent variables are also dimensionally independent and the following is straightforward. If not, then we have to one more step. Basically the idea is the dimensionally independent variables should determine  $[Q_o]$  and the other dimensionally dependent (dimensionally redundant) variables should appear in the form of dimensionless combination.

**EX:** Is  $\{\rho, v, f\}$  a physically independent set? Is it a dimensionally independent set?

**EX:**  $m, E_k, v$  are not physically independent nor dimensionally independent.

**EX:** Let's revisit the pendulum case.  $T = f(\ell, m)$ . It is clear here that the chosen variables are physically and dimensionally independent. Then the homogeneity dictates

$$[T] = [\ell]^x g^y = L^x L^y T^{-2y}.$$

Since  $[T] = T^{-1}$ , we solve

If  $k < n$ , find the combination of dimensionally independent variables for each dimensionally dependent variable following the similar process:

$$[Q_{k+1}] = [Q_1^{N_1} Q_2^{N_2} \cdots Q_k^{N_k}],$$

and so on. Then define a new dimensionless variable that would replace the dimensionally independent variable:

$$\Pi_{k+1} =$$

$$Q_1^{N_1} Q_2^{N_2} \cdots Q_k^{N_k}.$$

Finally,

$$Q_o = Q_1^\alpha Q_2^\beta \cdots Q_k^\kappa \Phi(\Pi_{k+1}, \dots, \Pi_n),$$

where the function  $\Phi$  is unknown at this point but can be determined through rigorous calculations. Note that  $\Phi$  is dimensionless!

**EX:** In the pendulum example, one project that the period of the pendulum should depend on  $m, g$ , and  $\ell$ . If you do DA with  $T = h(m, g, \ell)$ , where do you reach?

**EX:** When an object moves in air at a relatively high velocity, it will feel a drag force,  $f_d$ . One can choose that the drag force is related to the density of air  $\rho$ , the size of the object (diameter for a sphere)  $D$ , the speed of the object  $v$ , and also the sound speed of air  $c$ . The sound of speed may not be so obvious but let's take it for now.

- (1) Are they physically independent? *Yes, but not dimensionally independent*
- (2) Choose a dimensionally independent set.  $\{\rho, D, v\}$

(3) Do DA and find  $f_d = \rho^x D^y v^z \Phi(\Pi[1])$ .