Mathematics Self-Assessment for Physics Majors

Yoonseok Lee

The lack of mathematical sophistication is a leading cause of difficulty for students in Classical Mechanics and other upper level physics courses. An official pre-requisite of PHY3321 is PHY2048, PHY2049, and the math requirements include MAC 2311, 2312 and 2313 (Vector Calculus). These math courses together cover derivatives and integrals of trig and log functions, series and sequences, analytic geometry, vectors and partial derivatives and multiple integrals. We will casually be using math from all of these subjects. None of these should be completely unfamiliar to you. **Fluency in these math skills is a necessary but not sufficient condition for your success in all upper level physics courses.**

The following discussions and questions are grouped by subject and in approximate order of difficulty—easiest first. These are representative of the level of mathematics which is expected in this course. You should be very comfortable and fluent with mathematics at this level, at least through section G except for section D. Differential Calculus. Section E, on differential equations, is probably more difficult for you but important and useful. The answers to the questions are not always given. If you do not know if your answer is correct, then you are not comfortable with mathematics at this level. The importance of Section H cannot be overemphasized. You will use Taylor expansion over and over again as far as you are dealing with physics (**Trust me!**). If you understand the Taylor series in section H, then you are likely to find section I, on calculators, interesting and amusing. Don't be surprised if my discussions seem confusing at first: To understand math and physics often requires multiple, multiple readings and drills while working out algebraic details with paper and pencil in hand. Never ever try to go through math or physics problems with your eyes. You may think you understand the problem or the subject. But when a similar problem is given in an exam, you will feel that you have *seen* this before but cannot solve it. **This is why I often hear from many students "I studied very hard (with my eyes) but I do not perform well in exams."** Finally, Section **??** involves an ordinary differential equation that has an interesting application to radioactivity.

A. Algebra

Q Solve for x:

$$f(x) = ax^2 + bx + c = 0.$$

For what value of x is $f(x)$ a maximum or a minimum?

Q Make a sketch of the function $y(x)$ where $y = mx + b$ and where m and b are constants. What are the meanings of the constants m and b in terms of your sketch?

Q Factorize

$$(a^2 + 4ab + 4b^2) \qquad \text{and} \qquad (a^2 - 9b^2)$$

B. Vector Algebra

Q Let $\vec{A} = 1\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ and $\vec{B} = 4\hat{\imath} + 5\hat{\jmath} + 6\hat{k}$.

What is $|\vec{A}|$?

What is $\vec{A} \cdot \vec{B}$?

What is $\vec{A} \times \vec{B}$?

What is the cosine of the angle between $\vec{A}$ and $\vec{B}$?

Do you know

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} =?$$

Q If $\vec{A} \cdot \vec{C} = 10$ and the angle between $\vec{A}$ and $\vec{C}$ is $30°$, then what is the magnitude of $\vec{C}$?

The location of a point in three dimensional space can be represented by a position vector in Cartesian coordinates (x, y, x):

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}.$$

This is the vector connecting from the origin to the location whose magnitude (length) is $r = \sqrt{x^2 + y^2 + z^2}$, and its unit vector is $\hat{r} = \frac{\vec{r}}{r}$. Consider an infinitesimal displacement from (x, y, z) to $(x + dx, y + dy, z + dz)$. Then the infinitesimal displacement can be given by $d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}$. Again $|d\vec{r}| = \sqrt{dx^2 + dy^2 + dz^2}$.

Q What are $\vec{r} \cdot d\vec{r}$ and $\vec{r} \times d\vec{r}$?

C. Calculus

Q If x_0, v_0 and a are constants and

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2$$

then what is dx/dt? What is d^2x/dt^2? If $a < 0$, does the function $x(t)$ curve up or down? If x is negative when $t = 0$ and x is positive when t is very large: then for precisely which values of t is x positive?

Q Evaluate the derivative

$$\frac{d}{dt} A\cos(\omega t + \phi)$$

where A, ω and ϕ are constant.

Q If $f(x) = \frac{x}{\cos x}$, what is $\frac{df(x)}{dx}$?

Q If $f(x) = \tan(ax^2 + b)$, what is $\frac{df(x)}{dx}$?

Q Plot $y = \frac{1}{3}x^3 - 2x^2 + 3x + 2$ without using a graphing calculator. *By just looking at the functional form can you tell how many extrma at most exist in this curve?*

Q What is the value of x where the following function peaks (a is a positive constant)?

$$f(x) = \frac{1}{\sqrt{(a^2 - x^2)^2 + 4}}.$$

Can you plot the above function?

Q Evaluate the following integrals

$$\int_0^{\pi} \sin\theta \, d\theta,$$

$$\int_{-L/2}^{L/2} \frac{dz}{(z^2 + a^2)^{3/2}}$$

where L and a are constant,

$$\int \frac{k}{x^2} \, dx$$

where k is a constant, and

$$\int_1^x \frac{k}{x} \, dx.$$

Is $\sin x$ an odd or even function? How about $\cos x$, $\sin^2 x$, $x + 5x^5$?

If function $f(x)$ is even, then $f(x) = f(-x)$. If $f(x)$ is odd, then $f(x) = -f(-x)$. Do you see then why

$$\int_{-a}^{+a} f(x)dx = 0$$

for any odd function $f(x)$?

D. Differential Calculus

If $f = f(x)$ (f is a function of one variable x), then $\frac{df}{dx}$ would tell us how does f varies when the argument x change by an infinitesimal amount dx:

$$df = \left(\frac{df}{dx}\right) dx.$$

This implicitly means that infinitesimal change of a function (curve) can be given by a **linear variation** using the slope, $\frac{df}{dx}$, and the amount of change, dx. Extending to three dimension, for a function of three variables, $F(x, y, z)$ one can simply extend to

$$dF = \left(\frac{\partial F}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y}\right)dy + \left(\frac{\partial F}{\partial z}\right)dz,$$

using partial derivatives (derivative keeping other variables fixed). In Cartesian coordinate, one can express the above in a more compact form:

$$dF = \left(\frac{\partial F}{\partial x}\hat{x} + \frac{\partial F}{\partial y}\hat{y} + \frac{\partial F}{\partial z}\hat{z}\right) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z}) = \nabla F \cdot d\vec{r}.$$

∇F is called the **gradient** of F. It works as a vector and has its magnitude and direction. Therefore, you can write $\nabla F \cdot d\vec{r} = \mid \nabla F \mid\mid d\vec{r} \mid \cos\theta$. Then, what is the geometrical meaning of ∇F? *For a given variation $d\vec{r}$, ∇F points in the direction of maximum dF.* ∇F plays the same role as the slope in 1-D function, $\frac{df}{dx}$. Therefore, $\mid \nabla F \mid$ *gives the slope (rate of change) along the maximal direction.*

Q Can you show that $\nabla r = \hat{r}$ for $r = \sqrt{x^2 + y^2 + z^2}$?

Q Show that $\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$.

Q Calculate $\nabla f(x, y, z)$ for $f(x, y, z) = x^2yz^3$ and $f(x, y, z) = x^2 - y + z^3$.

One can also look at ∇F as a vector ∇ operated on a scalar F with $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$. Once this notation is adopted, one can expand the usage of the operator ∇ called **del**.

Divergence:

$$\nabla \cdot \vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (v_x\hat{x} + v_y\hat{y} + v_z y) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.$$

A dot product between two vectors results in a scalar. Simply applying this, $\nabla \cdot \vec{r} = 3$ where $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$. Geometrically, $\nabla \cdot \vec{v}$ is a measure of how much the vector $\vec{v}$ *diverges* from a point of interest. For example, $\nabla \cdot \vec{v}\mid_{x_o,y_o,z_o}$ reflects how much the vector field diverges from the point (x_o, y_o, z_o). Therefore, if $\vec{v} = \hat{x} + 2\hat{y}$, meaning constant vector all over the space, you can immediately see that anywhere in the space the vector field produces divergence.

Curl:

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

Q Write down each component of $\nabla \times \vec{v}$.

Q $\nabla \times \nabla F = 0$ for any scalar function $F(x, y, z)$?

Geometrically, $\nabla \times \vec{v}$ is a measure of the vector $\vec{v}$ winds around a point of interest.

E. Differential equations

These next two problems might be difficult or possibly unfamiliar to you, but take a careful look at them because these are very important in classical mechanics.

Q Find two different functions which satisfy the differential equation

$$\frac{d^2 f(x)}{dx^2} - \lambda^2 f(x) = 0.$$

Q Find two different functions which satisfy the differential equation

$$\frac{d^2 f(x)}{dx^2} + \omega^2 f(x) = 0.$$

F. Trigonometry and Geometry

Euler's identity,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

is probably new to you. But it provides a convenient and easy way to derive some of the basic trig identities such as

$$\begin{aligned} e^{i(\alpha+\beta)} &= e^{i\alpha}e^{i\beta} \\ \cos(\alpha+\beta) + i\sin(\alpha+\beta) &= (\cos\alpha + i\sin\alpha) \times (\cos\beta + i\sin\beta) \end{aligned}$$

or, after multiplying out the right hand side,

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta + i(\cos\alpha\,\sin\beta + \sin\alpha\,\cos\beta)$$

The real and the imaginary parts of this equation give the well known trig identities:

$$\cos(\alpha \pm \beta) = \cos\alpha\,\cos\beta \mp \sin\alpha\,\sin\beta$$

and

$$\sin(\alpha \pm \beta) = \sin\alpha\,\cos\beta \pm \cos\alpha\,\sin\beta.$$

or, after multiplying out the right hand side,

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta + i(\cos\alpha\,\sin\beta + \sin\alpha\,\cos\beta)$$

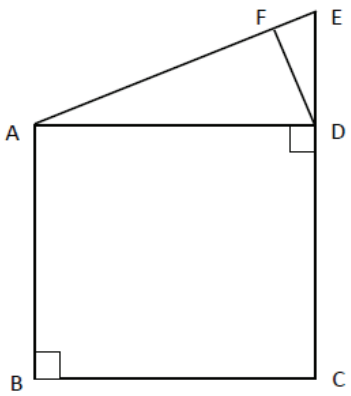

FIG. 1:

At this point let's check if you are comfortable with complex numbers. Any complex number z can be represented in one of the following ways:

$$\begin{aligned} z &= a + ib \quad (a, b \in \mathbf{R}) \\ &= |z|e^{i\theta} \quad (\theta = \arctan\frac{b}{a}) \\ &= |z|e^{i\delta}e^{i(\theta-\delta)} = Ce^{i\theta'} \quad (C \in \mathbf{C}). \end{aligned}$$

And,

$$z^* = a - ib = |z|e^{-i\theta} = C^*e^{-i\theta'},$$
$$|z| = \sqrt{zz^*}.$$

Q A real function $x(\theta)$ is given by

$$\begin{aligned} x(\theta) &= C_1e^{i\theta} + C_2e^{-i\theta} \quad (C_1, C_2 \in \mathbf{C}, \quad \text{and} \quad \theta \in \mathbf{R}) \\ &= B_1\cos\theta + B_2\sin\theta \quad (B_1, B_2 \in \mathbf{R}). \end{aligned}$$

Show that (i) $C_2 = C_1^*$ and (ii) $B_1 = C_1 + C_2$, $B_2 = i(C_1 - C_2)$.

Q Use the Euler identity to show that

$$\sin^2\theta + \cos^2\theta = 1.$$

Hint: start with $e^{i\alpha}e^{-i\alpha} = 1$ and then use the Euler Identity.

At this point, it is appropriate to introduce hyperbolic functions.

$$e^{\pm x} = \cosh x \pm \sinh x.$$

Q What is the first derivative of $\tanh x$?

Q See the figure above. $\overline{AB} = \overline{BC} = 2$, $\overline{DE} = 1$, and $\angle(DFE) = \pi/2$.

What is $\angle(FAD)$?
What is $\overline{AF}$?
What is $\overline{DF}$?
What is $\cos\theta$ ($\theta = \angle(EDF)$)?
Let's call the crossing point of $\overline{AD}$ and $\overline{BE}$ G. What is $\overline{DG}$?

G. Sums

Q Evaluate the sum

$$S(x) = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad |x| < 1 \ .$$

Ans: Note that

$$\begin{aligned} S(x) = \sum_{n=0}^{\infty} x^n &= 1 + \sum_{n=1}^{\infty} x^n \\ &= 1 + x\sum_{n=0}^{\infty} x^n \\ &= 1 + xS(x) \end{aligned}$$

So we have

$$\begin{aligned} S &= 1 + xS \\ (1-x)S &= 1 \\ \text{and, finally} \qquad S(x) &= \frac{1}{1-x}. \end{aligned}$$

By simply substituting $x = -a$, you have

$$\frac{1}{1+a} = 1 - a + a^2 - a^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n a^n.$$

Q How about the following summation?

$$S_N(x) = \sum_{n=0}^{N} x^n.$$

H. Taylor expansions of a function

Any differentiable function $f(x)$ may be approximated in the neighborhood of a point x_0 by the Taylor expansion

$$f(x) = f(x_0) + (x - x_0)\frac{df}{dx}_{x=x_0} + \frac{1}{2}(x-x_0)^2\frac{d^2f}{dx^2}_{x=x_0} + \cdots + \frac{1}{n!}(x-x_0)^n\frac{d^nf}{dx^n}_{x=x_0} + \cdots \quad (1)$$

For example, consider $f(x) = 1/(1-x)$, expanded about $x_0 = 0$. Then

$$\begin{aligned} f(x) &= 1/(1-x), \\ \frac{df}{dx}_{x=0} &= [1/(1-x)^2]_{x=0} = 1 \end{aligned}$$

$$\begin{aligned} \frac{d^2 f}{dx^2}_{x=0} &= 2[1/(1-x)^3]_{x=0} = 2 \\ \frac{d^3 f}{dx^3}_{x=0} &= 6[1/(1-x)^4]_{x=0} = 6 \\ \frac{d^n f}{dx^n}_{x=0} &= n![1/(1-x)^{n+1}]_{x=0} = n! \end{aligned}$$

The Taylor expansion for $1/(1-x)$ with $x_0 = 0$ is now

$$\frac{1}{1-x} = 1 + x + \frac{1}{2}x^2 \times 2 + \frac{1}{6}x^3 \times 6 + \cdots + \frac{1}{n!}x^n \times n! + \cdots$$

And this is easily seen to be

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

the same as our example for doing sums above!

Taylor expansions of this sort are extremely useful in physics. You will have to use Taylor expansion over and over again in physics. Trust me! For example in special relativity when we are interested to see how close special relativity is to Newtonian physics for small speeds v, we usually make the assumption that $v/c \ll 1$. Then we make Taylor expansions of the relevant formulae, and include only the terms proportional to v/c or maybe also v^2/c^2.

Q Try this! You do not have to understand physics here. Just follow the mathematical procedure. The displacement z of a particle of rest mass m_o, resulting from a constant force $m_o g$ along the z-axis is

$$z = \frac{c^2}{g}\{[1 + (\frac{gt}{c})^2]^{\frac{1}{2}} - 1\},$$

including relativistic effect. Find the displacement z as a power series in time t. Compare with the well-known classical result,

$$z = \frac{1}{2}gt^2.$$

Here, g is the gravitational acceleration and c is the speed of light.
Hint: You should realize $gt/c << 1$ and behave as a small parameter as ϵ in the formulae above. In the complete classical limit where the speed of light is considered infinite, you will recover the classical result. You know that you cannot just put $c = \infty$ in the above expression, which will give you meaningless $z = 0$.

Common Taylor expansions give approximations such as

$$\begin{aligned}\frac{1}{1-\epsilon} &= 1+\epsilon+O(\epsilon^2),\\ (1+\epsilon)^n &= 1+n\epsilon+O(\epsilon^2),\end{aligned}$$

$$\begin{aligned}\sqrt{1-\epsilon} &= 1-\frac{1}{2}\epsilon+O(\epsilon^2),\\ \frac{1}{\sqrt{1-\epsilon}} &= 1+\frac{1}{2}\epsilon+O(\epsilon^2),\end{aligned}$$

$$\begin{aligned}e^{\epsilon} &= 1+\epsilon+\frac{1}{2}\epsilon^2+\frac{1}{6}\epsilon^3+\frac{1}{24}\epsilon^4+O(\epsilon^5),\\ e^{i\epsilon} &= 1+i\epsilon-\frac{1}{2}\epsilon^2-i\frac{1}{6}\epsilon^3+\frac{1}{24}\epsilon^4+O(\epsilon^5),\end{aligned}$$

$$\begin{aligned}\ln(1+\epsilon) &= \epsilon-\frac{1}{2}\epsilon^2+\frac{1}{3}\epsilon^3+O(\epsilon^4),\\ \ln(1-\epsilon) &= -\epsilon-\frac{1}{2}\epsilon^2-\frac{1}{3}\epsilon^3+O(\epsilon^4).\end{aligned}$$

The $O(\epsilon^n)$ term here is standard mathematical notation to mean a function which is less than some constant times ϵ^n in the limit that $\epsilon \to 0$. In other words for small ϵ, $O(\epsilon^n)$ is no bigger than something times ϵ^n.

We can use the Euler identity $e^{i\epsilon} = \cos\epsilon + i\sin\epsilon$ to easily pick off the purely real terms from this last expansion which give the expansion of $\cos\epsilon$ for a small angle ϵ, and the purely imaginary terms, which give the expansion of $\sin\epsilon$ for small ϵ:

$$\begin{aligned}\cos\epsilon &= 1-\frac{1}{2}\epsilon^2+\frac{1}{24}\epsilon^4+O(\epsilon^6)\\ \text{and}\qquad \sin\epsilon &= \epsilon-\frac{1}{6}\epsilon^3+O(\epsilon^5).\end{aligned}$$

Since $\cos\epsilon$ *is an even function, you can only have terms of a even power such as* $\epsilon^0 = const$, $\epsilon^{2,4,\cdots}$. *Similarly for* $\sin\epsilon$ *only odd power terms exist.*

Let's go back to Eq.(1). If a function $f(x)$ has a extremum (maximum or minimum) at $x = x_o$, then

$$\frac{df}{dx}_{x=x_0} = 0.$$

This means that around $x = x_o$, the shape of the function is parabolic bending upward (minimum) if $\frac{d^2f}{dx^2}_{x=x_0} > 0$ or bending downward (maximum) if $\frac{d^2f}{dx^2}_{x=x_0} < 0$. *An analytic function can be approximated by a quadratic function near its minimum or maximum!*

Q Show that for $x << 1$, $\tanh x \approx x$, and $\tanh x \to \pm 1$ as $x \to \pm\infty$. Then, sketch the curve of $y = \tanh x$ on the graph.

I. Calculators

When solving a physics problem, think with your brain not with your calculator! Before touching your calculator, check to see that your algebraic answer has the correct units and that it has the expected behavior for various limits. It is nearly impossible to check the correctness of an answer once you touch your calculator. You might find it amusing that the number 10^{100} has been given the name googol, and 10^{googol} is called googolplex—and these names were coined well before the internet was invented. But, note the difference in spelling between googol and the name of the internet search engine. As an aside: The internet was invented by physicists who wanted to exchange easily experimental data between the United States and Europe.

Here are a couple examples which are relevant to one of the homework problems for this course. Let $f(n) = n^2$, where n is an integer. First evaluate $f(10^2) - f(10^2 - 1)$ on your calculator. You should get 199. Now try to evaluate $f(10^{100}) - f(10^{100} - 1)$. Your calculator will choke on this problem, but your brain can easily find the answer to 100 significant digits. Note that

$$f(n) - f(n-1) = n^2 - (n-1)^2 = n^2 - (n^2 - 2n + 1) = 2n - 1.$$

With $n = 10^{100}$, it is easy to see that $f(n) - f(n-1) = 2 \times 10^{100} - 1 \approx 2 \times 10^{100}$ with 100 significant digits.

Here is a second, more challenging, problem. Let $f(n) = n^{-2}$ where n is an integer. Evaluate $f(10^{100} - 1) - f(10^{100})$. Your calculator will also choke on this problem, but again you can easily find the answer to about 100 significant digits. Use the Taylor expansion

$$f(n + \delta n) = f(n) + \delta n \frac{df}{dn} + \ldots \Rightarrow f(n + \delta n) - f(n) = \delta n \frac{df}{dn} + \ldots = -\delta n \frac{2}{n^3} + \ldots$$

With $n = 10^{100}$ and $\delta n = -1$, we easily have $f(10^{100} - 1) - f(10^{100}) = 2 \times 10^{-300} + \ldots$, where the $\ldots$ represents terms which are comparable to $1/n^4 = 10^{-400}$ or smaller. Thus, the answer is correct for the first 100 digits.

For a final example which reveals the limitations of your calculator, evaluate

$$1 - \sqrt{1 - 3 \times 10^{-30}}$$

The answer is not zero. Analytically, find an approximation to the answer. In this context, the word "analytically" means that you should use algebra and calculus to find the answer. And you shouldn't touch a calculator or computer.
Hint: use a Taylor expansion.